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ON THE FRACTIONAL PARTIAL DERIVATIVE AND IT'S APPLICATION

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0. Abstract

There are many definitions of the fractional derivative. It is purpose of this paper to show some results which were got for fractional partial derivative of functions of two variables and to give an application of the fractional partial derivative.

1. On the fractional partial derivative

M. Riesz [1] and M.A. Bassam [2] gave the following definition for fractional partial derivative.

DEFINITION 1. The fractional partial derivative of functions $f(z,w)$ of two variables z and w is given by

$$D^{\alpha,\beta}_{g(z),\ h(w)} f(z,w)=\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{-4\pi^2}$$
$$\times\int_{g^{-1}(0)}^{(z^+)}\frac{g'(\xi)}{[g(\xi)-g(z)]^{\alpha+1}}\int_{h^{-1}(0)}^{(w^+)}\frac{f(\xi,\zeta)h'(\zeta)d\zeta d\xi}{[h(\zeta)-h(w)]^{\beta+1}}$$

where $Re(\alpha)$, $Re(\beta)<0$, $f(z,w)$, $g(z)$ and $h(w)$ are assumed to possess sufficient regularity to give the definition meaning, the notation

$$\int_{g^{-1}(0)}^{(z^+)}$$

on this integral implies that the contour of integration starts at $g^{-1}(0)$, enclosing singularities of f, and the notation $D^{\alpha,\beta}_{g(z),h(w)}\ f_{(z,w)}$ means the fractional derivative of $f(z,\mathrm{w})$ of order β with respect to $h(w)$ holding z fixed, followed by the derivative of order α with respect to $g(z)$ holding w fixed.

In this paper, the fractional partial derivative of functions $f(z,w)$ of two variables z and w is given as follows.

DEFINITION 2. The fractional partial derivative of functions $f(z,w)$ of two variables z and w is given by

$$D_{z,w}^{\alpha,\beta} f(z,w)=\frac{\partial^2}{\partial z\partial w}\left\{\frac{\Gamma(\alpha)\Gamma(\beta)}{-4\pi^2}\int_0^{(z+)}\frac{1}{(\xi-z)^\alpha}\int_0^{(w+)}\frac{f(\xi,\zeta)d\zeta d\xi}{(\zeta-w)^\beta}\right\},$$

where $0\leqq\alpha,\beta<1$ and $f(z,w)$ is assumed to possess sufficient regularity to give the definition meaning.

THEOREM I. *If the function* $f(z,w)$ *is analytic and* $0\leqq\alpha,\beta<1$, *then*

$$D_{z,w}^{\alpha,\beta} f(z,w)=\frac{1}{\Gamma(1-\alpha)\Gamma(1-\beta)}\int_0^z\frac{1}{(z-\xi)^\alpha}\int_0^w\frac{f''_{\xi,\zeta}(\xi,\zeta)d\zeta d\xi}{(w-\zeta)^\beta}.$$

PROOF. We have

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{-4\pi^2}\int_0^{(z+)}\frac{1}{(\xi-z)^\alpha}\int_0^{(w+)}\frac{f(\xi,\zeta)d\zeta d\xi}{(\zeta-w)^\beta}$$

$$=\frac{\Gamma(\alpha)\Gamma(\beta)}{-4\pi^2}\int_0^{(z+)}\frac{1}{(\xi-z)^\alpha}\left\{\int_0^w\frac{f(\xi,\zeta)d\zeta}{(\zeta-w)^\beta}+\int_0^{2\pi}\frac{f(\xi,re^{i\theta})re^{i\theta}id\theta}{(re^{i\theta}-w)^\beta}+\int_w^0\frac{e^{-2\pi i\beta}f(\xi,\zeta)d\zeta}{(\zeta-w)^\beta}\right\}d\xi$$

$$=\frac{\Gamma(\alpha)\Gamma(\beta)}{-4\pi^2}\int_0^{(z+)}\frac{1}{(\xi-z)^\alpha}(1-e^{-2\pi i\beta})\int_0^w\frac{f(\xi,\zeta)d\zeta d\xi}{(\zeta-w)^\beta}$$

$$=\frac{\Gamma(\alpha)}{2\pi i\Gamma(1-\beta)}\int_0^{(z+)}\frac{1}{(\xi-z)^\alpha}e^{-\pi i\beta}\int_0^w\frac{f(\xi,\zeta)d\zeta d\xi}{(\zeta-w)^\beta}$$

$$=\frac{\Gamma(\alpha)}{2\pi i\Gamma(1-\beta)}\int_0^{(z+)}\frac{1}{(\xi-z)^\alpha}\int_0^w\frac{f(\xi,\zeta)d\zeta d\xi}{(w-\zeta)^\beta}$$

$$=\frac{\Gamma(\alpha)}{2\pi i\Gamma(1-\beta)}\left\{\int_0^z\frac{1}{(\xi-z)^\alpha}\int_0^w\frac{f(\xi,\zeta)d\zeta d\xi}{(w-\zeta)^\beta}+\int_0^{2\pi}\frac{1}{(\rho e^{i\phi}-z)^\alpha}\int_0^w\frac{f(\rho e^{i\phi},\zeta)d\zeta\rho e^{i\phi}id\phi}{(w-\zeta)^\beta}+\int_z^0\frac{e^{-2\pi i\alpha}}{(\xi-z)^\alpha}\int_z^w\frac{f(\xi,\zeta)d\zeta d\xi}{(w-\zeta)^\beta}\right\}$$

$$=\frac{\Gamma(\alpha)}{2\pi i\Gamma(1-\beta)}(1-e^{-2\pi i\alpha})\int_0^z\frac{1}{(\xi-z)^\alpha}\int_0^w\frac{f(\xi,\zeta)d\zeta d\xi}{(w-\zeta)^\beta}.$$

$$=\frac{1}{\Gamma(1-\alpha)\Gamma(1-\beta)}e^{-\pi i\alpha}\int_0^z\frac{1}{(\xi-z)^\alpha}\int_0^w\frac{f(\xi,\zeta)d\zeta d\xi}{(w-\zeta)^\beta}$$

$$=\frac{1}{\Gamma(1-\alpha)\Gamma(1-\beta)}\int_0^z\frac{1}{(z-\xi)^\alpha}\int_0^w\frac{f(\xi,\zeta)d\zeta d\xi}{(w-\zeta)^\beta}$$

Hence, by means of definition 2,

$$D_{z,w}^{\alpha,\beta}f(z,w)=\frac{1}{\Gamma(1-\alpha)\Gamma(1-\beta)}\frac{\partial}{\partial z}\left[\int_0^z\frac{1}{(z-\xi)^\alpha}\frac{\partial}{\partial w}\left\{\int_0^w\frac{f(\xi,\zeta)d\zeta}{(w-\zeta)^\beta}\right\}d\xi\right]$$

$$=\frac{1}{\Gamma(1-\alpha)\Gamma(1-\beta)}\frac{\partial}{\partial z}\left[\int_0^z\frac{1}{(z-\xi)^\alpha}\frac{\partial}{\partial w}\left\{\left[\frac{(w-\zeta)^{1-\beta}f(\xi,\zeta)}{1-\beta}\right]_0^w\right.\right.$$

$$\left.\left.+\int_0^w\frac{(w-\zeta)^{1-\beta}f(\xi,\zeta)d\zeta}{1-\beta}\right\}d\xi\right]$$

$$=\frac{1}{\Gamma(1-\alpha)\Gamma(1-\beta)}\frac{\partial}{\partial z}\left[\int_0^z\frac{1}{(z-\xi)^\alpha}\frac{\partial}{\partial w}\left\{\int_0^w\frac{(w-\zeta)^{1-\beta}f(\xi,\zeta)d\zeta}{1-\beta}\right\}d\xi\right]$$

$$=\frac{1}{\Gamma(1-\alpha)\Gamma(1-\beta)}\frac{\partial}{\partial z}\left[\int_0^z\frac{1}{(z-\xi)^\alpha}\left[\frac{(w-\zeta)^{1-\beta}f(\xi,\zeta)}{1-\beta}\right|_{\zeta=0}\right.$$

$$\left.\left.+\int_0^w\frac{\partial}{\partial w}\left\{\frac{(w-\zeta)^{1-\beta}f'_\zeta(\xi,\zeta)}{1-\beta}\right\}d\zeta\right]d\xi\right]$$

$$=\frac{1}{\Gamma(1-\alpha)\Gamma(1-\beta)}\frac{\partial}{\partial z}\left\{\int_0^z\frac{(z-\xi)^{1-\alpha}}{1-\alpha}\int_0^w\frac{f'_\zeta(\xi,\zeta)d\zeta d\xi}{(w-\zeta)^\beta}\right\}$$

$$=\frac{1}{\Gamma(1-\alpha)\Gamma(1-\beta)}\left[\frac{(z-\xi)^{1-\alpha}}{1-\alpha}\int_0^w\frac{f'_\zeta(\xi,\zeta)d\zeta}{(w-\zeta)^\beta}\right|_{\xi=0}$$

$$\left.+\int_0^z\frac{\partial}{\partial z}\left\{\frac{(z-\xi)^{1-\alpha}}{1-\alpha}\int_0^w\frac{f''_{\zeta,\xi}(\xi,\zeta)d\zeta}{(w-\zeta)^\beta}\right\}d\xi\right]$$

$$=\frac{1}{\Gamma(1-\alpha)\Gamma(1-\beta)}\left[\int_0^z\frac{\partial}{\partial z}\left\{\frac{(z-\xi)^{1-\alpha}}{1-\alpha}\int_0^w\frac{f''_{\xi,\zeta}(\xi,\zeta)d\zeta}{(w-\zeta)^\beta}\right\}d\xi\right]$$

$$=\frac{1}{\Gamma(1-\alpha)\Gamma(1-\beta)}\frac{\partial}{\partial z}\left\{\int_0^z\frac{1}{(z-\xi)^\alpha}\int_0^w\frac{f'_\zeta(\xi,\zeta)d\zeta d\xi}{(w-\zeta)^\beta}\right\}=\frac{1}{\Gamma(1-\alpha)\Gamma(1-\beta)}$$

$$\times\frac{\partial}{\partial z}\left[\left[\frac{(z-\xi)^{1-\alpha}}{1-\alpha}\int_0^w\frac{f'_\zeta(\xi,\zeta)d\zeta d\xi}{(w-\zeta)^\beta}\right]_0^z+\int_0^z\frac{(z-\xi)^{1-\alpha}}{1-\alpha}\int_0^w\frac{f'_\zeta(\xi,\zeta)d\zeta d\xi}{(w-\zeta)^\beta}\right]$$

$$=\frac{1}{\Gamma(1-\alpha)\Gamma(1-\beta)}\int_0^z\frac{1}{(z-\xi)^\alpha}\int_0^w\frac{f''_{\xi,\zeta}(\xi,\zeta)d\zeta d\xi}{(w-\zeta)^\beta}$$

2. An application of the fractional partial derivative

Let the notation $f*g(z,w)$ denote the convolution product of $f(z,w)$ and $g(z,w)$, and the notation $LL[f(z,w)]$ denote the double Laplacian transformation of $f(z,w)$. Then, the following formulae were shown by M. P. Humbert [3].

(1) $LL[f(z,w)]=\int_0^\infty\int_0^\infty e^{-sz-\sigma\omega}f(z,w)dzdw,$

(2) $LL[f*g(z,w)]=LL[f(z,w)]LL[g(z,w)],$

(3) $LL\left\{\frac{\partial}{\partial z}f(z,w)\right\}=sLL[f(z,w)],$

and

(4) $LL\left\{\frac{\partial}{\partial w}f(z,w)\right\}=\sigma LL[f(z,w)].$

THEOREM 2. *If the function $f(z,w)$ is analytic and $0\leqq\alpha,\beta<1$, then we have*

$$LL[D_{z,w}^{\alpha,\beta}f(z,w)]=s^\alpha\sigma^\beta LL[f(z,w)].$$

PROOF. From theorem 1, we have

$$D_{z,w}^{\alpha,\beta}f(z,w)=\frac{1}{\Gamma(1-\alpha)\Gamma(1-\beta)}\frac{\partial}{\partial z}\left[\int_0^z\frac{1}{(z-\xi)^\alpha}\frac{\partial}{\partial w}\left\{\frac{f(\xi,\zeta)d\zeta}{(w-\zeta)^\beta}\right\}d\xi\right]$$

$$=\frac{1}{\Gamma(1-\alpha)\Gamma(1-\beta)}\frac{\partial}{\partial z}\left[z^{-\alpha}*\frac{\partial}{\partial w}\left\{w^{-\beta}*f(z,w)\right\}\right].$$

Taking the double Laplacian transformation of both sides, then using the formulae (2), (3) and (4),

$$LL[D^{\alpha,\beta}_{z,w}f(z,w)]=\frac{1}{\Gamma(1-\alpha)\Gamma(1-\beta)}LL\left[\frac{\partial}{\partial z}\left[z^{-\alpha}*\frac{\partial}{\partial w}\left\{w^{-\beta}*f(z,w)\right\}\right]\right]$$

$$=\frac{1}{\Gamma(1-\alpha)\Gamma(1-\beta)}\left[sLL\left[z^{-\alpha}*\frac{\partial}{\partial w}\left\{w^{-\beta}*f(z,w)\right\}\right]\right.$$

$$\left.-\left[z^{-\alpha}*\frac{\partial}{\partial w}\left\{w^{-\beta}*f(z,w)\right\}\right]\Big|_{z=+0}\right]$$

$$=\frac{1}{\Gamma(1-\alpha)\Gamma(1-\beta)}sLL(z^{-\alpha})LL\left[\frac{\partial}{\partial w}\left\{w^{-\beta}*f(z,w)\right\}\right]$$

$$=\frac{1}{\Gamma(1-\beta)}s^{\alpha}\left[\sigma LL\left\{w^{-\beta}*f(z,w)\right\}-\left\{w^{-\beta}*f(z,w)\right\}\Big|_{w=+0}\right]$$

$$=\frac{1}{\Gamma(1-\beta)}s^{\alpha}\sigma LL(w^{-\beta})LL[f(z,w)]=s^{\alpha}\sigma^{\beta}LL[f(z,w)].$$

Therefore, theorem 2 is established.

COROLLARY. *If the function $f(z,w)$ is analytic and $0\leqq\alpha,\beta<1$, then*

$$LL[D^{\alpha}_{z}f(z,w)]=s^{\alpha}LL[f(z,w)]$$

and

$$LL[D^{\beta}_{w}f(z,w)]=\sigma^{\beta}LL[f(z,w)].$$

THEOREM 3. *If the function $f(z,w)$ is analytic and $0\leqq\alpha,\beta<1$, then*

$$D^{\alpha,\beta}_{z,w}f(z,w)=\frac{1}{\Gamma(-\alpha)\Gamma(-\beta)}\int_0^z\frac{1}{(z-\xi)^{1+\alpha}}\int_0^w\frac{f(\xi,\zeta)d\zeta d\xi}{(w-\zeta)^{1+\beta}}.$$

PROOF. From theorem 2,

$$D^{\alpha,\beta}_{z,w}f(z,w)=L^{-1}L^{-1}\left\{s^{\alpha}\sigma^{\beta}LL[f(z,w)]\right\}$$

$$=L^{-1}L^{-1}\left[LL\left\{\frac{1}{\Gamma(-\alpha)\Gamma(-\beta)}z^{-1-\alpha}w^{-1-\beta}\right\}LL[f(z,w)]\right]$$

$$=\frac{1}{\Gamma(-\alpha)\Gamma(-\beta)}\int_0^z\frac{1}{(z-\xi)^{1+\alpha}}\int_0^w\frac{f(\xi,\zeta)d\zeta d\xi}{(w-\zeta)^{1+\beta}}.$$

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