Kyungpook Math. J.
Volume 20, Number 1
June, 1980

ON NEWTON'S METHOD IN COMPLETE FIELDS

By Vichian Laohakosol

G. Bachman in his book [1] gave an algorithm for determining roots of polynomials over non-archimedean valued fields similar to the well-known Newton's algorithm in the real case. His two main theorems are as follows:

Let F be a field completed with respect to a non-archimedean valuation $|\,|$; also let V be its associated valuation ring, i.e.

$$V=\{v\in F\ ;\ |v|\leq 1\}.$$

THEOREM 1. *Let $P(x)=x^n+p_{n-1}x^{n-1}+\cdots+p_0$ be a polynomial with coefficients in V. If there exists an $a_1\in F$ such that*

$$|P(a_1)|<1 \qquad \text{and} \qquad |P'(a_1)|=1,$$

then the sequence

$$(1)\qquad \begin{cases} a_2=a_1-P(a_1)/P'(a_1),\\ a_3=a_2-P(a_2)/P'(a_2),\\ \quad\vdots \end{cases}$$

converges to a root $a\in V$ of $P(x)$.

THEOREM 2. *Let $P(x)$ be as in theorem 1. If there exists an $a_1\in F$ such that*

$$|P(a_1)|<1,\ \ P'(a_1)\neq 0,\ \ |P'(a_1)|\leq 1$$

and

$$|P(a_1)/P'(a_1)^2|<1,$$

then the sequence (1) converges to a root $a\in V$ of $P(x)$.

In this paper, we employ this same Bachman's method to slightly generalise the above two theorems to cover a larger class of polynomials and give one example to illustrate this.

THEOREM 1G. *Let $P(x)=x^n+p_{n-1}x^{n-1}+\cdots+p_0$ be a polynomial with coefficients in F and let*

$$M=\max(|p_0|,\ |p_1|,\dots,|p_{n-1}|,\ 1).$$

If there exists an $a_1\in V$ such that

$$|P(a_1)|<1/M \qquad \text{and} \qquad |P'(a_1)|=1,$$

then the sequence (1) *converges to a root* $a\in V$ *of* $P(x)$.

PROOF. Since $P(x)$ is a polynomial, then by Taylor's series expansion, we have

$$P(x+h)=P(x)+hP'(x)+h^2g(x, h),$$

where

$$g(x, h)=\frac{1}{2!}P^{(2)}(x)+\frac{h}{3!}P^{(3)}(x)+\cdots+\frac{h^{n-2}}{n!}P^{(n)}(x),$$

then

(2) $$P(a_2)=P(a_1)-\frac{P(a_1)}{P'(a_1)}P'(a_1)+\left(\frac{P(a_1)}{P'(a_1)}\right)^2 g\left(a_1, \frac{-P(a_1)}{P'(a_1)}\right)$$

where

$$g\left(a_1, \frac{-P(a_1)}{P'(a_1)}\right)=\frac{1}{2!}P^{(2)}(a_1)-\frac{1}{3!}\frac{P(a_1)}{P'(a_1)}P^{(3)}(a_1)+\cdots.$$

Since the coefficients of the polynomials $P^{(i)}(x)/i!$ $(i=2, 3,\cdots)$ all have valuations $\leq M$, $|a_1|\leq 1$ and $|P(a_1)/P'(a_1)|<1/M$, then

$$\left|g\left(a_1, \frac{-P(a_1)}{P'(a_1)}\right)\right|\leq M.$$

Also from (2),

(3) $$|P(a_2)|\leq|P(a_1)|^2M<1/M.$$

Using Taylor's series expansion again, we obtain

$$P'(a_2)=P'(a_1)-\frac{P(a_1)}{P'(a_1)}f\left(a_1, \frac{-P(a_1)}{P'(a_1)}\right),$$

where

$$f\left(a_1, \frac{-P(a_1)}{P'(a_1)}\right)=\frac{P^{(2)}(a_1)}{2!}+\left(\frac{-P(a_1)}{P'(a_1)}\right)\frac{P^{(3)}(a_1)}{3!}+\cdots.$$

The same reasoning as for the function g yields

$$\left|f\left(a_1, \frac{-P(a_1)}{P'(a_1)}\right)\right|\leq M,$$

and so

(4) $$|P'(a_2)|=1.$$

Because of (3) and (4) we see that a_2 satisfies the same assumptions as those of a_1; therefore, the procedure can be repeated successively and we get

$$|a_2-a_1|=|P(a_1)|,$$
$$|a_3-a_2|=|P(a_2)|\leq|P(a_1)|^2M,$$
$$|a_4-a_3|=|P(a_3)|\leq|P(a_1)|^4M^3,$$
$$\vdots$$
$$|a_{n+1}-a_n|=|P(a_n)|\leq|P(a_1)|^{2^{n-1}}M^{2^{n-1}-1}$$
$$<|P(a_1)|(|P(a_1)|M)^{2^{n-1}-1}\to 0 \quad (n\to\infty).$$

Upon putting

$$a=a_1+(a_2-a_1)+(a_3-a_2)+\cdots=\lim a_n,$$

we see that

$P(a)=\lim P(a_n)=0$ and the proof is complete.

The generalisation of theorem 2 can also be obtained similarly and since no new idea is involved, we merely state the result without proof:

THEOREM 2G. *Let $P(x)$ and M be as in theorem 1G. If there exists an $a_1\in V$ such that*

$$|P(a_1)|<1/M,\ P'(a_1)\neq 0,\ |P'(a_1)|\leq 1$$

and

$$|P(a_1)/P'(a_1)^2|<1/M,$$

then the sequence (1) *converges to a root $a\in V$ of $P(x)$.*

EXAMPLE. Let p be a rational prime, $F=Q_p$ be the field of p-adic numbers, $V=Z_p$ be the ring of p-adic integers. Consider $P(x)=x^3+x^2/p+x+p^2$ so that $M=p$. Taking $a_1=0$; then we see that $|P(0)|=p^{-2}<1/M$, $|P'(0)|=1$. In this example, theorem 1G is applicable while theorem 1 is not.

Department of Mathematics,
Kasetsart University,
Bangkok, Thailand.

REFERENCE

[1] Bachman, G., *Introduction to p-adic numbers and valuation theory*, Academic Press, New York and London, 1964.