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SOME FIXED POINT THEOREMS FOR DENSIFYING MAPPINGS

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1. Introduction

The concept of the measure of non-compactness of a bounded set in a metric space was introduced by Kuratowski [5]. In [4], Furi and Vignoli used this notion to prove a fixed point theorem for continuous densifying mappings which are weakly contractive with respect to a lower semi-continuous function. Later on, Thomas [7] gave an improvement of the fixed point theorem of Furi and Vignoli by placing conditions on the iterates of the mapping rather than on the mappings itself. In this note, we introduce the concepts of pairwise-densifying mappings and pairwise-weakly contractive mappings with respect to two lower semi-continuous functions, and establish a result on fixed points for such mappings which includes both the above theorems as particular cases. To achieve our goal, we first prove a fixed point theorem in compact space. This subsidiary result generalizes fixed point theorems of Edelstein and Bailey.

We begin by introducing the following notations and definitions. Throughout this paper, let X denote a set, and S and T self-mappings of X.

DEFINITION 1.1. Let A be a bounded set in X. Then the ***measure of non-compactness***, $\alpha(A)$, of A is the infimum of all $\varepsilon>0$, such that A admits a finite covering by sets of diameter less than ε. Some important properties of the measure of non-compactness are listed below:

(i) $\alpha(A)=0 \Longleftrightarrow A$ is precompact.

(ii) $\alpha(A)=\alpha(\bar{A})$, $\bar{A}$ stands for the closure of A.

(iii) $\alpha(A\cup B)=\max\{\alpha(A), \alpha(B)\}$, where B is a bounded subset of X.

DEFINITION 1.2. (cf. [4]). The mapping T is said to be ***densifying*** if for every bounded subset A of X, $\alpha(T(A))<\alpha(A)$ whenever $\alpha(A)>0$.

A contraction mapping and a completely continuous mapping are examples of densifying mappings.

DEFINITION 1.3. (cf. [7]). Let F be a real-valued lower semi-continuous function defined on $X\times X$. The mapping T is said to ***iteratively weakly F-contractive*** at the point x if there exists an integer $n(x)$ such that

$$F(T^{n(x)}(x), T^{n(x)}(y)) < F(x, y) \text{ for all } y \in X \text{ with } y \neq x.$$

We introduce the following more general notions:

DEFINITION 1.4. Mappings S and T are said to be *pairwise-densifying* if for every bounded subset A of X, with $\alpha(A) > 0$

$$\alpha(TS^2T(A)) < \alpha(A).$$

Clearly, two densifying mappings S and T will be always pairwise-densifying.

DEFINITION 1.5. Let F_1 and F_2 be real-valued lower semi-continuous functions on $X \times X$. The mappings S and T are said to be *pairwise-weakly contractive with respect to* F_1 and F_2 at the point x if

(i) $$F_1(TSx, STy) < F_2(x, y)$$

and

(ii) $$F_2(STx, TSy) < F_1(x, y),$$

for all $y \in X$ such that $y \neq x$.

For $S = T$ (respectively S = identity), we simply say that T^2(respectively T) is weakly contractive with respect to F_1 and F_2.

2. Main rsults

Firstly, we prove a fixed point theorem for pairwise-weakly contractive mappings defined on a compact space X. This result is then used to prove our main result for pairwise-densifying mappings.

PROPOSITION 2.1. *Let X be compact such that ST is continuous and S, T are pairwise-weakly contractive with respect to F_1 and F_2 at all points of X. Then ST or TS has a fixed point. Further, a common fixed point of ST and TS is always unique.*

PROOF. Define $\phi : X \to R^+$ (non-negative reals) by $\phi(x) = F_1(x, STx)$. Then ϕ is a lower semi-continuous function on the compact set X, and therefore attains a minimum value at some point ξ of X. Suppose that $\xi \neq ST(\xi)$ and $TS(ST(\xi)) \neq ST(\xi)$. Then we have

$$\begin{aligned}
\phi(TS^2T(\xi)) &= F_1(TS(ST(\xi)),\ ST(TS^2T(\xi))) \\
&< F_2(ST(\xi), TS(ST(\xi))) \\
&< F_1(\xi, ST\xi) \\
&= \phi(\xi).
\end{aligned}$$

This contradicts of minimality of ξ. Hence either $\xi=ST\xi$ or $TS(ST\xi)=ST\xi$

Thus ST has a fixed point ξ or TS has a fixed point $ST\xi$. Unicity of a common fixed point of ST and TS is clear from definition 1.5. This completes the proof.

COROLLARY 2.2. *Let T be a mapping of a compact set X into itself with T^2 continuous and also T^2 is weakly contractive with respect to F_1 and F_2 at all points of X. Then T has a unique fixed point.*

PROOF. That T^2 has a unique fixed point, say ξ, follows immediately from proposition 2,1 by taking $S=T$. Now $T^2(T\xi)=T(T^2\xi)=T\xi$. So $T\xi$ is also a fixed point of T^2. Uniquencess of ξ ends the proof.

COROLLARY 2.3. *Let T be a self-mapping on a compact set X with T^2 continuous. Let F be a lower semi-continuous function on $X\times X$ such that for all x and y in X:*

$$F(T^2x,\ T^2y)<F(x,y),\quad x\neq y.$$

Then T has a unique fixed point.

COROLLARY 2.4. *Let T be a continuous mapping on the compact set X such that T is weakly contractive with respect to F_1 and F_2. Then T has a unique fixed point.*

COROLLARY 2.5. *Let T be a continuous mapping of the compact set X into itself such that for a lower semi-continuous function F on $X\times X$ and for all x, y in X with $x\neq y$:*

$$F(Tx,\ Ty)<F(x,y)$$

Then T has a unique fixed point.

REMARK. If F is a metric on X, then corollary 2.5 reduces to a theorem of Edelstein [3].

COROLLARY 2.6. *Let T be a continuous mapping on a compact metric space (X,d), and for all x, y in X with $x\neq y$, there exists a positive integer $n(x,y)$ such that*

$$(*)\qquad d(T^nx,\ T^ny)<d(x,y).$$

Then T has a unique fixed point.

PROOF. Let $S=T^{n-1}$. Then $ST=TS=T^n$ is continuous from (*).

The result now follows from proposition 2.1, where d is playing the role of F_1 and F_2.

REMARK. Corollary 2.6 resembles corollary 2 of theorem 1 or Bailey [1]. We can also deduce a result like corollary 3 of Bailey [1] from our corollary 2.6.

Now we shall use proposition 2.1 to prove the following:

THEOREM 2.7. *Let S and T be pairwise-densifying mappings on a complete metric space (X, d) into itself such that ST and TS are continuous. Further, let S and T be pairwise-weakly contractive with respect to F_1 and F_2. If for some $x_0 \in X$, the sequence $\{x_n\}$ defined by $ST(x_{2n}) = x_{2n+1}$, $TS(x_{2n+1}) = x_{2n+2}$ for $n = 0, 1, 2\cdots$, is bounded, then either ST or TS has a fixed point. Further, a common fixed point of ST and TS is always unique.*

RROOF. Consider the set

$$A = \bigcup_{n=0}^{\infty} \{x_{2n}\}.$$

Then

$$TS^2T(A) = \bigcup_{n=0}^{\infty} \{x_{2n+2}\},$$

whence $TS^2T(A) \subset A$, and the continuity of ST and TS imply that $TS^2T(\overline{A}) \subset \overline{A}$. We shall now prove that $\overline{A}$ is compact. Since X is complete, it suffices to show that $\alpha(A) = 0$. Suppose that $\alpha(A) > 0$. Then by noting that

$$A = \{x_0\} \cup (TS^2T(A)),$$

we have

$$\begin{aligned} \alpha(A) &= \alpha(\{x_0\} \cup (TS^2T(A))) \\ &= \max\{\alpha\{x_0\}, \alpha(TS^2T(A))\} \\ &= \alpha(TS^2T(A)) \\ &< \alpha(A), \end{aligned}$$

which is a contradiction. Hence it follows that $\overline{A}$ is compact. Since $TS^2T(\overline{A}) \subset \overline{A}$, the rest of the proof is now identical with that of the proposition 2.1.

COROLLARY 2.8. *Let T be a self-mapping of a complete metric space (X, d)*

such that T^2 is continuous and T^4 is densifying. Further, let T^2 be weakly contractive with respect to F_1 and F_2. If for some $x_0 \in X$, the sequence $\{x_n\}$ defined by $T^2(x_n) = x_{n+1}$ for $n = 0, 1, 2, \cdots$, is bounded, then T has a unique fixed point.

COROLLARY. 2.9. *Let T be a continuous self-mapping on a complete metric space (X, d) such that T^2 is densifying. Further, let T be weakly contractive with respect to F_1 and F_2. Then if some sequence of iterates starting from $x_0 \in X$ is bounded, T has a unique fixed point.*

REMARK. If $F_1 = F_2$, and T is densifying then corollary 2.9 becomes a fixed point theorem of Furi and Vignoli [4].

COROLLARY 2.10. *Let T be a self-mapping of a complete metric space such that T^n is densifying for some fixed integer n, T is iteratively weakly F-contractive at all points in X, and $T^{n(x)}$ is continuous at x. If some sequence of iterates starting from $x_0 \in X$ is bounded, then T has a unique fixed point.*

PROOF. Let $S = T^{n-1}$. Then $ST = TS = T^n$ is continuous. Since T^n is densifying, $TS^2T = (T^n)^2$ is also densifying. Then by taking $F_1 = F_2 = F$, it follows from theorem 2.7, that T^n has a unique fixed point. Hence T has a unique fixed point. This completes the proof.

REMARKS. Corollary 2.10 is the main result of Thomas [7], and also theorem 2(I) of Lee [6]. From corollary 2.10 we can also deduce corollary 2.1(I) of Lee [6] and a theorem of Bryant [2].

3. Problem and example

Let X denote a non-empty set, and P a non-empty family of pseudo-metrics on X such that the collection $\{S_p(x, r) : x \in X,\ p \in P,\ r \in (0, +\infty)\}$ forms a base for a Hausdorff topology on X, where $S_p(x, r)$ is the open sphere of p-radius r about x. The concepts related to a topology for X to be concerned, will be those related to the topology generated by P. Also, S and T denote self-mappings on X. Now we pose the following problem:

Can we prove a result analogous to theorem 2.7, when X is sequentially complete with respect to P, and for all $x, y \in X$, $x \neq y$ and for all $p \in P$

$$p(STx, TSy) < p(x,y), \quad p(x,y) > 0$$
$$= 0, \quad p(x,y) = 0.$$

REMARK. Should we solve the above problem, we can deduce theorem 2(II) and other related results of Lee [6] as corollaries to this result.

Finally, we give an example to justify the statement of proposition 2.1.

EXAMPLE. Consider a set $X=\{a, b\}$ with discrete metric $F=F_1=F_2$. Define mappings S and T on X into itself by

$$Sa=a, \quad Sb=a, \quad Ta=a, \quad Tb=b.$$

Then S and T are continuous mappings on the compact set X, and also conditions of proposition 2.1 are satisfied. This example shows that the mappings S and T in proposition 2.1 are not necessarily identical. Also, the mappings S and T can possibly have other fixed points although a common fixed point has to be unique.

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