Kyungpook Math. J.
Volume 20, Number 2
December, 1980

ALMOST POINTWISE PERIODIC SEMIGROUPS

By Younki Chae

The author investigated a structure theorem of a pointwise periodic semigroup on an arc [1]. In this paper, a structure theorem of an almost pointwise periodic semigroup on an arc is given. The results obtained are:

(1) A compact semigroup S is almost pointwise periodic if and only if for each compact subset K of S, $K^2 \subset K$ implies $K^2 = K$.

(2) Every almost pointwise periodic semigroup on an arc is a semilattice.

1. Introduction

A topological semigroup is a Hausdorff space with a continuous associative multiplication, denoted by juxtaposition [2], [3]. Throughout, a semigroup will mean a topological semigroup. An arc is a continuum with exactly two non-cutpoints. It is well known that any arc admits a total order and has one non-cutpoint as a least element and the other non-cutpoint as a greatest element [4]. It is supposed that an arc to have such a total order on it. We will denote an arc with end points a and b, $a<b$, by $[a,b]$ and if $x,y \in [a,b]$, $x<y$, then

$$[x,y]=\{t \mid x \leq t \leq y\},\quad \langle x,y\rangle = \{t \mid x<t<y\}.$$

A standard thread is a semigroup on an arc in which the greatest element is an identity and the least element is a zero. The real unit interval $[0,1]$ under the ordinary multiplication is called the *real thread*, and the real interval $[1/2,1]$ under the multiplication

$$xy=\max\left\{\frac{1}{2},\ \text{ordinary product of } x \text{ and } y\right\}$$

is called the *nil thread*.

An element e of a semigroup is called an *idempotent* iff $e^2=e$. If a semigroup S has a zero z, then $x \in S$ is called a *nilpotent* of S iff $x^n=z$ for some positive integer n.

Note that every element of the nil thread except 1 is a nilpotent.

The following lemma gives the structure of standard threads which will be found in [2].

LEMMA 1.1. *Let S be a standard thread and let E be the set of all idempotents of S. If $\langle e,f\rangle$ is a component of $S-E$, then $[e,f]$ is iseomorphic to either the real thread or the nil thread.*

2. Almost pointwise periodic semigroups

DEFINITION. A semigroup S is termed *almost pointwise periodic at* $x\in S$ iff for each open set U about x, there is an integer $n>1$ such that $x^n\in U$.

S is said to be *almost pointwise periodic* iff S is almost pointwise periodic at every $x\in S$ [5].

LEMMA 2.1. *Let K be a compact subsemigroup of a semigroup S. Then S is not almost pointwise periodic at every point of $K-K^2$.*

PROOF. Since K is compact and since the binary operation in S is continuous, K^2 is compact. Let $x\in K-K^2$. Then there is an open set U about x such that $U\cap K^2=\phi$. Now since $K^n\subset K^2$ $(n\geq 2)$, $x^n\in K^2$ $(n\geq 2)$. This shows that $\{x^2, x^3, \cdots\}\cap U=\phi$, i.e., S is not almost pointwise periodic at x.

THEOREM 2.2. *A compact semigroup S is almost pointwise periodic iff for each compact subset K of S, $K^2\subset K$ implies $K^2=K$.*

PROOF. Suppose S is almost pointwise periodic and let K be a compact subset of S such that $K^2\subset K$. If $K^2\neq K$, by lemma 2.1, S is not almost pointwise periodic at each point of $K-K^2$. This contradicts the hypothesis and hence $K^2=K$. Now suppose the condition holds. Assume that S is not pointwise periodic at a point $a\in S$. Then there is an open set U about a such that

$$U\cap\{a^2, a^3, \cdots\}=\phi,$$

i.e., $x\notin\{a^2, a^3,\cdots\}^*$ (the closure of $\{a^2, a^3,\cdots\}$). Let us set

$$P=\{a^2, a^3,\cdots\}^*, \quad K=P\cup\{a\}.$$

Since S is compact, K is a compact subset of S. By the compactness of P, one obtain $P^2=(\{a^2, a^3, \cdots\}\{a^2, a^3, \cdots\})^*=\{a^4, a^5, \cdots\}^*\subset P$.

Then $K^2=P^2\cup aP\cup Pa\cup\{a^2\}\subset P=K-\{a\}$.

This shows that $K^2\subset K$ and $K^2\neq K$ which contradicts the assumption. Hence S is almost pointwise periodic.

COROLLARY 2.3. *Every closed ideal of a compact almost pointwise periodic semigroup is full* [1].

THEOREM 2.4. *Every almost pointwise periodic standard thread is a semilattice.*

PROOF. Let S be an almost pointwise periodic standard thread and let $\langle e, f\rangle$ be a component of $S-E$, where E is the set of all idempotents of S. By lemma 1.1, $[e, f]$ is iseomorphic to the real thread or the nil thread.

Suppose $[e,f]$ is iseomorphic to the real thread. Let $a\in\langle e, f\rangle$. Then $a^n<a^2$ $(n=3, 4, \cdots)$. Since S is Hausdorff, there are open sets (b,c) and (p,q) about a^2 and a respectively such that $(b, c)\cap(p, q)=\phi$. Hence we have $\{a^2, a^3, \cdots\}\cap(p,q)=\phi$.

This contradicts the fact that S is almost pointwise periodic. Now suppose $[e,f]$ is iseomorphic to the nil thread. Then every element of $\langle e,f\rangle$ is a nilpotent of $[e,f]$. Let $x\in\langle e,f\rangle$. Then there is the least positive integer m such that $x^m=e$. Let U_j be an open set about x such that $x^j\notin U_j$ $(j=2, 3, \cdots, m)$ and let $U=\cap\{U_j \mid j=2, 3, \cdots, m\}$. Then $x\in U=U^0$, $\{x^2, x^3, \cdots, x^m\}\cap U=\phi$.

If $p>m$, since $x^p=x^m x^{p-m}=ex^{p-m}=e$, there is an open set V about x such that $x^p=e\notin V$. Let $W=U\cap V$. Then $x\in W=W^0$, $\{x^2, x^3, \cdots\}\cap W=\phi$.

This is a contradiction since S is almost pointwise periodic. Hence E is dense in S. Since E is closed, we have $S=E$, i.e., S is a semilattice.

Kyungpook National University

REFERENCES

[1] Chae, Y., *On pointwise periodic semigroups and full ideals*, Kyungpook Math. J. 15(1975).

[2] Paalman-De Miranda, A.B., *Topological semigroups*, Mathematical centre tracts. Vol. 11, Amsterdam 1964.

[3] Wallace, A.D., *The structure of topological semigroups*, Bull. Amer. Math. Soc. 61(1955).

[4] Hocking, J.G. and Young, G.S., *Topology*, Addison-Wesley, Massachusetts 1961.

[5] Whyburn, G.T., *Analytic topology*, Amer. Math. Soc. Coll. Publs. Vol. 28, New York 1942.