ALMOST POINTWISE PERIODIC SEMIGROUPS

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The author investigated a structure theorem of a pointwise periodic semigroup on an arc [1]. In this paper, a structure theorem of an almost pointwise periodic semigroup on an arc is given. The results obtained are:

1. A compact semigroup $S$ is almost pointwise periodic if and only if for each compact subset $K$ of $S$, $K^2 \subseteq K$ implies $K^2 = K$.

2. Every almost pointwise periodic semigroup on an arc is a semilattice.

1. Introduction

A topological semigroup is a Hausdorff space with a continuous associative multiplication, denoted by juxtaposition [2], [3]. Throughout, a semigroup will mean a topological semigroup. An arc is a continuum with exactly two non-cutpoints. It is well known that any arc admits a total order and has one non-cutpoint as a least element and the other non-cutpoint as a greatest element [4]. It is supposed that an arc to have such a total order on it. We will denote an arc with end points $a$ and $b$, $a < b$, by $[a, b]$ and if $x, y \in [a, b]$, $x < y$, then

$$[x, y] = \{t | x \leq t \leq y\}, \langle x, y \rangle = \{t | x < t < y\}.$$ 

A standard thread is a semigroup on an arc in which the greatest element is an identity and the least element is a zero. The real unit interval $[0, 1]$ under the ordinary multiplication is called the real thread, and the real interval $[1/2, 1]$ under the multiplication

$$xy = \max \left\{ \frac{1}{2}, \text{ ordinary product of } x \text{ and } y \right\}$$

is called the nil thread.

An element $e$ of a semigroup is called an idempotent iff $e^2 = e$. If a semigroup $S$ has a zero $z$, then $x \in S$ is called a nilpotent of $S$ iff $x^n = z$ for some positive integer $n$.

Note that every element of the nil thread except 1 is a nilpotent.

The following lemma gives the structure of standard threads which will be found in [2].
LEMMA 1.1. Let $S$ be a standard thread and let $E$ be the set of all idempotents of $S$. If $\langle e, f \rangle$ is a component of $S - E$, then $[e, f]$ is isomorphic to either the real thread or the nil thread.

2. Almost pointwise periodic semigroups

DEFINITION. A semigroup $S$ is termed almost pointwise periodic at $x \in S$ iff for each open set $U$ about $x$, there is an integer $n > 1$ such that $x^n \in U$.

S is said to be almost pointwise periodic iff $S$ is almost pointwise periodic at every $x \in S$.

LEMMA 2.1. Let $K$ be a compact subsemigroup of a semigroup $S$. Then $S$ is not almost pointwise periodic at every point of $K - K^2$.

PROOF. Since $K$ is compact and since the binary operation in $S$ is continuous, $K^2$ is compact. Let $x \in K - K^2$. Then there is an open set $U$ about $x$ such that $U \cap K^2 = \phi$. Now since $K^2 \subseteq K^2$ ($n \geq 2$), $x^n \in K^2$ ($n \geq 2$). This shows that $(x^2, x^3, \ldots) \cap U = \phi$, i.e., $S$ is not almost pointwise periodic at $x$.

THEOREM 2.2. A compact semigroup $S$ is almost pointwise periodic iff for each compact subset $K$ of $S$, $K^2 \subseteq K$ implies $K^2 = K$.

PROOF. Suppose $S$ is almost pointwise periodic and let $K$ be a compact subset of $S$ such that $K^2 \subseteq K$. If $K^2 \neq K$, by lemma 2.1, $S$ is not almost pointwise periodic at each point of $K - K^2$. This contradicts the hypothesis and hence $K^2 = K$. Now suppose the condition holds. Assume that $S$ is not pointwise periodic at a point $a \in S$. Then there is an open set $U$ about $a$ such that $U \cap (a^2, a^3, \ldots) = \phi$.

i.e., $x \notin (a^2, a^3, \ldots)$ (the closure of $(a^2, a^3, \ldots)$). Let us set

$P = (a^2, a^3, \ldots)^*$, $K = P \cup \{a\}$.

Since $S$ is compact, $K$ is a compact subset of $S$. By the compactness of $P$, one obtain $P^2 = ((a^2, a^3, \ldots)(a^2, a^3, \ldots))^* = (a^4, a^5, \ldots)^* \subseteq P$.

Then $K^2 = P^2 \cup aP \cup Pa \cup \{a^3\} \subseteq P = K - \{a\}$.

This shows that $K^2 \subseteq K$ and $K^2 \neq K$ which contradicts the assumption. Hence $S$ is almost pointwise periodic.

COROLLARY 2.3. Every closed ideal of a compact almost pointwise periodic semigroup is full [1].

THEOREM 2.4. Every almost pointwise periodic standard thread is a semilattice.
PROOF. Let $S$ be an almost pointwise periodic standard thread and let $\langle e, f \rangle$ be a component of $S-E$, where $E$ is the set of all idempotents of $S$. By lemma 1.1, $[e, f]$ is isomorphic to the real thread or the nil thread.

Suppose $[e, f]$ is isomorphic to the real thread. Let $a \in \langle e, f \rangle$. Then $a^n < a^2 \ (n = 3, 4, \ldots)$. Since $S$ is Hausdorff, there are open sets $(b, c)$ and $(p, q)$ about $a^2$ and $a$ respectively such that $(b, c) \cap (p, q) = \emptyset$. Hence we have $(a^2, a^3, \ldots) \cap (p, q) = \emptyset$.

This contradicts the fact that $S$ is almost pointwise periodic. Now suppose $[e, f]$ is isomorphic to the nil thread. Then every element of $\langle e, f \rangle$ is a nilpotent of $[e, f]$. Let $x \in \langle e, f \rangle$. Then there is the least positive integer $m$ such that $x^m = e$. Let $U_j$ be an open set about $x$ such that $x \in U_j \ (j = 2, 3, \ldots, m)$ and let $U = \cap (U_j) \ (j = 2, 3, \ldots, m)$. Then $x \in U = U^0 = \{x^2, x^3, \ldots, x^m\} \cap U = \emptyset$.

If $p > m$, since $x^p = x^m x^{p-m} = e x^{p-m} = e$, there is an open set $V$ about $x$ such that $x^p = e \in V$. Let $W = U \cap V$. Then $x \in W = W^0 = \{x^2, x^3, \ldots\} \cap W = \emptyset$.

This is a contradiction since $S$ is almost pointwise periodic. Hence $E$ is dense in $S$. Since $E$ is closed, we have $S=E$, i.e., $S$ is a semilattice.

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REFERENCES