ON STRUCTURES OF LEFT BIPOTENT NEAR RINGS

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1. Introduction

Throughout this paper $N$ will mean a zero-symmetric near ring (that is, $a0=0$ for all $a\in N$) with or without identity, which satisfies the right distributive law. $N$ is said to be left bipotent if $Na=Na^2$ for all $a\in N$, and an $s$-near ring if $a\in Na$ for all $a\in N$ ([6]). These types of near rings were introduced by J.L. Jat and S.C. Choudhary.

For convenience, we quote their results.

THEOREM 1.1. If $N$ is left bipotent, the following are equivalent.

1. $N$ is an $s$-near ring
2. $N$ has no non-zero nilpotent elements and
3. $N$ is regular (i.e. for any $a\in N$, there exists $a'\in N$ such that $aa'a=a$)

On the other hand, Howard E. Bell proved in [7] that

THEOREM 1.2. If $N$ has no non-zero nilpotent elements, it is isomorphic to a subdirect sum of near rings with no non-zero divisors of zero.

In this paper we investigated structures of left bipotent near rings and some elementary properties of near rings.

For undefined terminologies, we refer to [8].

2. Results

LEMMA 2.1. Homomorphic images of left bipotent $s$-near rings are also such.

PROOF. Let $f: N\rightarrow N'$ be a homomorphism of near rings $N$ onto $N'$, and let $N$ be a left bipotent $s$-near ring. If $a\in N'$, there exists $b\in N$ such that $f(b)=a$. By assumption, we have $Nb=Na^2$. Then $f(Nb)=f(N)f(b)=N'a$ and $f(Nb^2)=f(N)f(b)^2=N'a^2$. Thus $N'a=N'a^2$. Now since $b\in Nb$, we have $a=f(b)\in f(Nb)=N'a$.

PROPOSITION 2.2. Let $N$ be a left bipotent $s$-near ring. Then it is isomorphic
PROOF. In view of Theorem 1.1, 1.2, and Lemma 2.1, we may assume that each \( N_i \) is a left bipotent s-near ring with no non-zero divisors of zero.

For each \( i \), if \( N_i \) has a non-zero distributive element, then by [6] it is a near field. Now if \( N_i \) has no such an element, for any \( 0 \neq r \in N_i \), there exists \( r' \in N_i \) such that \( rr' = r \) by Theorem 1.1. Putting \( e = rr' \), \( e \) is an idempotent. Then for each \( x \in N_i \), we have \((xe-x)r = xee - exr = xrr - xr = 0\). Since \( N_i \) has no nonzero divisors of zero, we have \( xe = x \). Hence \( e \) is a right identity for \( N_i \).

We observe that homomorphic images of distributively generated (d.g.) near rings are d.g.. Thus a d.g. left bipotent s-near ring is isomorphic to a subdirect sum of near fields.

**COROLLARY 2.3.** Let \( N \) be left bipotent with 1. \( N \) is isomorphic to a subdirect sum of near fields.

**PROOF.** Since an epimorphism carries identity to identity, each subdirect summand of \( N \) endowed with an identity. Obviously an identity is a distributive element, so the proof is immediately established by Proposition 2.2.

**COROLLARY 2.4.** A left bipotent near ring with 1 has commutative addition.

**PROOF.** By Corollary 2.3, the proof is trivial.

**PROPOSITION 2.5.** A left bipotent near ring with 1 is a ring iff it is d.g..

**PROOF.** \((\Rightarrow)\) Clear.

\((\Leftarrow)\) With the aid of Corollary 2.4, we need only the left distributive law. Let \( a, b, c \in N \) and put \( a = a_1 + \cdots + a_n \), where each \( a_i \) is distributive. Then we have

\[
a(b+c) = (a_1 + \cdots + a_n)(b+c)
= a_1(b+c) + \cdots + a_n(b+c)
= a_1b + a_1c + \cdots + a_nb + a_nc
= (a_1b + \cdots + a_nb) + (a_1c + \cdots + a_nc)
= (a_1 + \cdots + a_n)b + (a_1 + \cdots + a_n)c
= ab + ac.
\]
By a left annihilator of \( r \in N \), we mean the set
\[
I(r) = \{ x \in N | xr = 0 \}
\]
It is immediate that \( I(S) = \bigcap_{s \in S} I(s) \) for any non-empty subset \( S \) of \( N \). Analogously we may define right annihilators.

It is easily seen that left annihilators are left ideals and left annihilators of \( N \)-subgroups are ideals. But right annihilators are only closed under multiplications of elements of \( N \) on the right hand side.

Hereafter, for any non-empty subsets \( A \) and \( B \) of \( N \),
\[
AB = \{ ab | a \in A \text{ and } b \in B \}
\]

**Proposition 2.6.** If \( N \) has non-zero nilpotent elements, \( I(S) \) is an ideal for every non-empty subset \( S \) of \( N \).

**Proof.** We need only to show that \( I(S)N \subseteq I(S) \). Let \( x \in I(S) \) and \( s \in S \). Then \( xs = 0 \) implies \((sx)^2 = s(xs)x = 0 \). Hence \( sx = 0 \) by assumption. For any \( r \in N \), \((xr)^2 = xr(xr)r = 0 \) implies \( xr = 0 \). Thus \( xr \in I(S) \). Hence \( xr \in I(S) = \bigcap_{s \in S} I(s) \). Therefore \( I(S)N \subseteq I(S) \).

In view of Theorem 1.1, we immediately have

**Corollary 2.7.** In a left bipotent \( s \)-near ring, left annihilators are ideals.

**Proposition 2.8.** Let \( B \) be a minimal \( N \)-subgroup of \( N \), then either \( B^2 = 0 \) or there exists \( e^2 = e \in B \) such that \( B = Ne \).

**Proof.** If \( B^2 \neq 0 \), there exists \( 0 \neq b \in B \) such that \( Bb \neq 0 \). Since \( Bb \) is an \( N \)-subgroup and \( 0 \neq Bb \subseteq B \), \( Bb = B \). Now let \( I(b) = \{ r \in N | rb = 0 \} \) be the left annihilator of \( b \) in \( N \). Then \( I(b) \cap B = 0 \).

Now \( eb = b \) for some \( 0 \neq e \in B \), \( e^2 = eb \). Thus \( (e^2 - e)b = 0 \). Therefore \( e^2 - e \in I(b) \cap B = 0 \), so \( e^2 = e \). Thus we have \( B = Ne \) by the minimality of \( B \).

Again by Theorem 1.1, we have

**Corollary 2.9.** In a left bipotent \( s \)-near ring, every minimal \( N \)-subgroup of \( N \) has the form \( Ne \) for some \( e^2 = e \in N \).

**Lemma 2.10** If \( N \) has 1 and every \( N \)-subgroup of \( N \) is finitely generated, then \( N \) has the maximum condition on \( N \)-subgroups.
PROOF. Let \( A_1 \subset A_2 \subset \cdots \) be a chain of \( N \)-subgroups of \( N \). We set \( A = \bigcup_{i=1}^{\infty} A_i \). Then \( A \) is an \( N \)-subgroup of \( N \). Now let \( \{ a_1, \ldots, a_k \} \subset A \) be a generating set of \( A \). Then \( \{ a_1, \ldots, a_k \} \subset A_n \) for some \( n \in \mathbb{Z}^+ \). Hence \( A \subset A_n \subset A \). Therefore \( A_n = A_{n+1} = \cdots \) as required.

In a d.g. near ring, it is known that an \( N \)-subgroup of \( N \) is a left ideal iff it is a normal subgroup of \( (N, +) \).

**Lemma 2.11.** Let \( N \) be d.g. and let \( A, B \) be left ideals of \( N \), then \( A + B \) is also a left ideal.

**Proof.** Since \( A \) and \( B \) are normal subgroup of \( (N, +) \), we see that \( A + B = B + A \) is also a normal subgroup of \( (N, +) \). Now it suffices to show that \( A + B \) is an \( N \)-subgroup of \( N \).

Let \( r \in N \). We may put \( r = r_1 + \cdots + r_n \), where each \( r_i \) distributive or anti-distributive. For any element \( a + b \in A + B \),

\[
\begin{align*}
    r(a + b) &= (r_1 + \cdots + r_n)(a + b) \\
    &= r_1(a + b) + \cdots + r_n(a + b).
\end{align*}
\]

For each \( 1 \leq i \leq n \), if \( r_i \) is distributive, \( r_i(a + b) = r_ia + r_ib \in A + B \), and if \( r_i \) is anti-distributive, \( r_i(a + b) = r_ib + r_ia \in B + A = A + B \). In any way \( r_i(a + b) \in A + B \) for \( 1 \leq i \leq n \). Therefore \( r(a + b) \in A + B \) as required.

**Proposition 2.12.** Let \( N \) be d.g. with 1. If every additive subgroup of \( N \) is normal then \( N \) has the maximum condition on \( N \)-subgroups iff every \( N \)-subgroup of \( N \) is finitely generated.

**Proof.** (\( \Leftarrow \)) It is already shown in Lemma 2.10.

(\( \Rightarrow \)) Under the hypothesis, the two concepts \( N \)-subgroup and left ideal coincide. So we turn the proof on left ideals.

Let \( A \) be a left ideal of \( N \). Consider all finitely generated left ideals of \( N \) contained in \( A \). By assumption, the set contains a maximal element, say, \( B \).

Let \( a \in A \) and consider \( B + Na \). By assumption and Lemma 2.11, \( B + Na \) is a left ideal and \( B \subset B + Na \subset A \). Then \( B = B + Na \) by the maximality of \( B \). Hence \( a \in B \) and consequently \( A = B \).
REFERENCES


