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A NOTE ON LEVITZKI RADICAL OF NEAR-RING

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Levitzki [5] has studied locally nilpotent rings to define the radical of a general ring. In this note we extend it to the case of near-rings. This provides a natural example of an F-radical to support the work of Scott [7] and Saxena and Bhandari [6]. It is shown that unlike rings, the locally nilpotent radical of a near-ring need not contain every locally nilpotent one side ideal. An example is given to show that the result of Herstein; "the set of all nilpotent elements of a ring R form an ideal if $(xy-yx)^n=0$ for all x, y in R and for some fixed positive integer n" does not hold for near-rings. A result of Amitsur [3] for rings is extended for a class of distributively generated near-rings.

An algebraic system $N=(N,+,\cdot,0)$ is called a *near-ring* if (i) $(N,+,0)$ is a group, (ii) $(N,\cdot)$ is a semi-group, (iii) $a\cdot(b+c)=a\cdot b+a\cdot c$ for all a,b,c in N and (iv) $0\cdot a=0$ for all a in N. A subset I of N is said to be an *ideal* of N if (i) $(I,+)$ is a normal subgroup of $(N,+)$ (ii) $(a+x)b-ab$ belongs to I for all a,b in N and x in I and (iii) $ax\in I$ for all $a\in N$ and $x\in I$. If (i) and (ii) are satisfied then I is called a *right ideal* of N. For other elementary properties of near-rings we refer to [7].

A near-ring N is said to be *locally nilpotent* if every finite subset of N is nilpotent. A distributively generated (d.g.) near-ring N is locally nilpotent if and only if subnear-ring generated by every finite subset of N is nilpotent. For, let S be a generating set of N, whose every finite subset is nilpotent and let $F=\{a_1, a_2, \cdots, a_n\}$ be a finite subset of N. Then the set $P=\{\pm s|s$ is in S and appears in the the representation of some $a_i\in F\}$ is a finite subset of S, and hence $P^m=(0)$ for some natural number m. We observe that $[F]$, the subring generated by F is $\bigcup_{i=0}^{\infty} S_i$ where $S_0=FU-F$, $S_n=\{x : x$ is in N and is a finite sum of finite products of elements from $S_{n-1}\}$, $n=1, 2, 3, \cdots$. It is easy to see that $S_i^m=0$ for all $i=1, 2, \cdots$, and hence $[F]^m=0$. The converse is obvious. However the two statements need not be equivalent for arbitrary near-rings. Analogous to rings [3] it is easy to see that a near-ring N is locally nilpotent, if and only if I and N/I are locally nilpotent for some ideal I of N. Thus the sum of two locally nilpotent ideals is locally nilpotent. Therefore $L(N)$, the sum of all locally nilpotent ideal containing every other locally nilpotent ideal with $L(N/L(N))=(0)$. Hence we have

THEOREM 1. *The class of all locally nilpotent near-rings is a hereditary radical class.*

$L(N)$ is called the *Levitzki radical* of N. For any universal class C, local nilpotence gives rise to a C-formation radical [6]. In fact, the class $\{(A, N) \mid A$ is locally nilpotent ideal of N, $N \in C\}$ is a C-formation radical class. The proof of the following characterization of $L(N)$ in terms of prime ideals is similar to that for rings [3].

THEOREM 2. *If N is a near-ring, then $L(N)=\bigcap\{P \mid P$ is a prime ideal with $L(N/P)=(0)\}$.*

As an immediate corollary, we have the following:

THEAREM 3. *Every near-ring N with $L(N)=(0)$ is isomorphic to a subdirect sum of prime near-rings with Levitzki Radical* (0).

The relationship of locally nilpotent radical with various other radicals is given by the chain.

$S(N) \subset l(N) \subset L(N) \subset U(N) \subset J_0(N) \subset D(N) \subset J_1(N) \subset J_2(N)$ where $S(N)$ is the sum of all nilpotent ideals of N, $l(N)$ is the intersection of all prime ideals of N, called *lower nil radical*, $U(N)$ is the sum of all nil ideals of N, called *upper nil radical*, and $D(N)$, $J_0(N)$, $J_1(N)$, $J_2(N)$ are as defined in [1]. If I is a nil ideal of a near-ring N satisfying $ba=0$ whenever $ab=0$ for all a, b in I, then $ab=0$ implies $axb=0$ for all a, b in I, x in N and hence $ax_1ax_2 \cdots ax_{n-1}a=0$ for $x_1, x_2, \cdots, x_{n-1}$ in N whenever $a^n=0$, a in I. Thus if $F=\{a_1, a_2, \cdots, a_n\}$ is a finite subset of N, then $F^{m_0}=0$, where $m_0=(m-1)n+1$, $m=\max\{n_1, n_2, \cdots, n_n\}$, n_i being the index of nilpotency of a_i. We have thus proved:

PROPOSTION 4. *If N is a near-ring satisfying $ab=0$ implies $ba=0$ for all a, b in N, then $L(N)=U(N)$.*

It is observed that for a d. g. near-ring the condition given in Proposition 1 can be sharpened by $s_1 s_2=0$ implies $s_2 s_1=0$ for all s_1, s_2 in S. Another sufficient condition for the equality of $L(N)$ and $U(N)$ is as follows:

PROPOSITION 5. *If N is a near-ring with identity satisfying d.c.c. on finitely generated nil right N-subgroup then* $L(N)=U(N)$.

PROOF. Let I be a nil right N-subgroup of N-subgroup of N and let J be any finite subset of I. For $n=1, 2, 3, \cdots$, let $\bar{J}_n$ denote the right N-subgroup of N generated by $J^n=\{a_1 a_2 \cdots a_n | a_i \in J\}$. We observe that $\bar{J}_1 \supset \bar{J}_2 \supset \bar{J}_3 \supset \cdots$ is a descending chain of finitely generated nil right N-subgroups and hence, $\bar{J}_m = \bar{J}_{m+1} \cdots = \bar{J}_{2m}$ for some m. If $\bar{K}=J_m \neq (0)$, and $\bar{K}_2$ is the right N-subgroup generated by K^2, then $K \supset \bar{K}_2 \supset K^2 \supset J^m, J^m=J^{2m}$, and hence $0 \neq K \supset \bar{K}_2 \supset \bar{J}_{2m}=\bar{J}_m=K$ giving $K_2 \neq (0)$, $K=\bar{K}_2$. By Zorn's Lemma, there exists a non zero minimal N-subgroup M generated by finite number of elements of I, and $M \cdot K \neq (0)$. Thus $m \cdot K \neq (0)$ for some m in M. If $(m \cdot K) \cdot K=(0)$, then $K \cdot K \subseteq$ Ker θ, where $\theta: N \rightarrow N$ is a homomorphism defined by $\theta(x)=mx$ for all x in N, and hence $K=\bar{K}_2 \subset$ Ker θ, contradicting $m \cdot K \neq (0)$. Thus $(m \cdot K) \cdot K \neq (0)$, and therefore $m \cdot K=M$ by minimality of M. Since $k \in K \subset I$, $m=mk=mk^2=\cdots=0$. This contradiction proves that $K=\bar{J}_m=(0)$, and hence $J^m=(0)$. Thus every finite subset of I is nilpotent, and hence I is locally nilpotent.

It is seen that for a near ring N satisfying d. c. c. on nil N-subgroups, $S(N) =l(N)=U(N)=L(N)$. In this case J is replaced by I in the proof of Proposition 2. Unlike rings, Levitzki radical $L(N)$ of a near-ring N need not contain every locally nilpotent one sided ideal.

EXAMPLE 1. Consider $G=(S_9, +)$, the symmetric group over 9 symbols. Let T be the near-ring generated by all inner automorphisms of G. Then $(T, +, 0)$ is a finte simple *d. g.* near ring with identity. For each x in T define $\theta_x: T \rightarrow T$ by $\theta_x(y)=xy$ for all y in T. Then θ_x is an endomorphism. Let N be the *d. g.* near-ring generated by the set $\{\theta_x | x$ is in $T\}$. Then N is a finite *d. g.* near-ring with identity and $L(N)=(0)$ [4]. But the intersection of all maximal right ideals of N is a nonzero locally nilpotent right ideal, which is not a two sided ideal [4].

Locally nilpotence of rings has a further interesting feature (given by Amitsur): If there exists a fixed positive integer n such that $x^n=0$ for all x in the ring, then the ring is locally nilpotent. The corresponding result holds for d. g. near-rings with $n=2$, and for a class of d. g. near-rings with $n=3$.

THEOREM 6. *If N is a d. g. near-ring with $x^2=0$ for all x in N, then N is locally nilpotent.*

PROOF. Let S be a multiplicative semigroup generating N. For s_1, s_2 in S,

$$0=(s_1+s_2)^2=s_1^2+s_2s_1+s_1s_2+s_2^2=s_2s_1+s_1s_2.$$

Thus

(1) $s_2s_1=-s_1s_2$ for all s_1, s_2 in S.

It can be easily shown that (1) holds for all s_1, s_2 in $SU-S$. If $F=\{x_1, x_2, \cdots, x_m\}$ is a finite subset of N, then $x_i=\sum_{j=1}^{r_i} t_{ij}$; t_{ij} in $SU-S$ for all i, j. Hence in view of (1), $F^{m_0}=(0)$ where $m_0=r_1+r_2+\cdots+r_m+1$. Thus N is locally nilpotent.

THEOREM 5. *Let N be a d.g. near-ring with generating set S. If $x^3=0$ for all x in N and $s^2=0$ for all s in S, then N is locally nilpotent.*

PROOF. For s_1, s_2 in S, $0=(s_1+s_2)^3=(s_1^2+s_2s_1+s_1s_2+s_2^2)\ (s_1=s_2)=s_1s_2s_1+s_2s_1s_2$.

Therefore

(2) $s_1s_2s_1=-s_2s_1s_2$ for all s_1, s_2 in S.

It is easy to varify that the relation (2) holds for all s_1, s_2 in $SU-S$. We now make an observation that any product of the form $y=\cdots s_1\cdots s_1\cdots$, where s_1 is in $SU-S$ and . represents some element of $SU-S$, is zero. For, by (2),

$$\begin{aligned} y&=\cdots s_1(s_{r_1}s_{r_2}\cdots s_{r_j})s_1\cdots \\ &=-[\cdots(s_{r_1}s_{r_2}\cdots s_{r_j})s_1(s_{r_1}s_{r_2}\cdots s_{r_j})\cdots] \\ &=-[\cdots s_{r_1}s_{r_2}(s_{r_3}s_{r_4}\cdots s_1s_{r_1})s_{r_2}\cdots] \\ &=\cdots s_{r_1}(s_{r_3}s_{r_4}\cdots s_1s_{r_1})s_{r_2}(s_{r_3}\cdots s_1s_{r_1})\cdots \\ &=-[\cdots s_{r_1}(s_{r_4}\cdots s_{r_j}s_1s_{r_1}s_{r_2})s_{r_3}(s_{r_4}\cdots s_{r_3})\cdots] \end{aligned}$$

Hence, repeating this process $j+1$ times leaving first s_{r_1} fixed, we get

$$y=\pm[\cdots s_{r_1}s_{r_1}\cdots]=0.$$

Now, let $P=[x_1, x_2, \cdots, x_m]$ be a finite subset of R. Each x_i is of the form $x_i=\sum_{j=1}^{n_i} s_{i_j}$ s_{i_j} in $SU-S$ for all i, j. Then $P^n=0$ with $n=r_1+r_2\cdots+r_m+1$. Hence R is locally nilpotent.

In a similar way, it can be proved that a d. g. near-ring N satisfying $x^n=0$ for all x in N, n fixed positive integer, and (2), is locally nilpotent. Herstein has proved the following result.

THEOREM ([3]). *If there exists a fixed positive integer n such that $(xy-yx)^n=0$ for all x, y in N, then the set of all nilpotent elements of N is an ideal.*

This need not hold for near-rings.

EXAMPLE 2. Let $N=(S_3, +)$ be the symmetric group on three elements say $N=\{0, a, x, 2x, a+x, x+a \mid +$ is the composition, $a=(1, 2)$, $x=(1, 2, 3)\}$. Define multiplication $\cdot$ in N by

$$z_1 \cdot z_2 = \begin{cases} z_2, & \text{if } z_1=a+x; \\ 0, & \text{otherwise.} \end{cases}$$

Then N is a near-ring ([2]) and $(uv-vu)^2=0$ for all $u, v \in N$. But the set of nilpotent elements, $\{0, a, x+a, x, 2x\}$, is not an ideal of N.

PROPOSITION 7. *If R is a locally nilpotent simple d. g. near-ring, then $R^3=(0)$.*

PROOF. Let $R^3 \neq (0)$. Then there exists x in R such that$R \times R \neq (0)$. Consider $\langle R \times R \rangle = \{\Sigma_f(-m_i+a_i x s_i+m_i);\ a_i, m_i$ are in R, s_i in $SU-U\}$, where S generates R. Then $(0) \neq \langle R \times R \rangle$ is a two sided ideal of R generated by $R \times R$. Hence $\langle R \times R \rangle = R$. So $x=\sum_{i=1}^{k}(-m_i+a_i x s_i+m_i)$ for some m_i, a_i in R and s_i in $SU-S$.

Consider $F=\{a_i, s_i \mid$ where a_i, s_i appear in the representation of $x\}$. Certainly F is a finite subset of R and so $F^n=(0)$ for some positive integer n.

Now,

$$\begin{aligned} x &= \sum_{i=1}^{k}(-m_i+a_i x s_i+m_i) \\ &= \sum_{i=1}^{k}(-m_i+a_i(\sum_{i=1}^{k}(-m_i+a_i x s_i+m_i))s_i+m_i) \\ &= \sum_{i=1}^{k}(-m_i^{(1)}+a_i^{(1)} x s_i^{(1)}+m_i^{(1)});\ \text{where } a_i^{(1)},\ s_i^{(1)} \text{ are in } F^2 \\ &= \sum_{i=1}^{k}(-m_i^{(n)}+a_i^{(n)} x\, s_i^{(n)}+m_i^{(n)});\ \text{where } a_i^{(n)},\ s_i^{(n)} \text{ are in } F^n \\ &= 0. \end{aligned}$$

Hence $R=(0)$. Thus $R^3=(0)$.

COROLLARY 8. If R is a simple d.g. near-ring which is locally nilpotent, then R^2 can not be an ideal of R. (For simple near-rings we assume $R^2 \neq (0)$.

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