ON DIRECT INJECTIVE MODULES

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1. Introduction

The concept of direct injectivity was introduced by W.K. Nicholson([3]). We know that injective modules are direct injective and its converse is not true in general. For instance, the $\mathbb{Z}$-module $\mathbb{Z}_4$ does not imply the direct injectivity, for the $\mathbb{Z}$-module $\mathbb{Z}$ is not direct injective but quasi-injective.

In this paper, we investigate properties of direct injective modules. Throughout this paper $M$ always denotes an $R$-module.

2. Results

DEFINITION 2.1. An $R$-module $M$ is said to be direct injective if and only if given direct summand $D$ of $M$ with injection $1_D : D \rightarrow M$ and a monomorphism $k : D \rightarrow M$, there exists $f \in \text{End}(M)$ such that $f \circ k = 1_D$.

PROPOSITION 2.2. For an $R$-module $M$, $M$ is direct injective if and only if for submodules $A,B$ of $M$, $B$-direct summand of $M$ and any monomorphism $\Phi : M/B \rightarrow A$, there exists $\Psi \in \text{Hom}_R(M,A)$ with $\Psi \circ \Phi = \nu$, where $\nu : M/B \rightarrow M$ is the canonical injection.

PROOF. Assume that $M$ is direct injective. Suppose $B$ is a direct summand of $M$. Then for canonical injection $\nu : M/B \rightarrow M$ and a monomorphism $\Phi : M/B \rightarrow A$, there exists a homomorphism $g : M \rightarrow M$ such that $g \circ i \circ \Phi = \nu$. Put $\Psi = g \circ i$, then $\Psi$ is the required. The converse implication is quite obvious.

The proof of the following proposition is similar to that of proposition 2.3. in [4].

PROPOSITION 2.3. A direct summand of a direct injective module is direct injective.

PROPOSITION 2.4. If $\alpha M$ is a direct summand of $M$ for each $\alpha \in \text{End}(M)$, then $M$ is direct injective.

REMARK. The reverse of the above proposition 2.4. is not true in general.
Let $f : \mathbb{Z}_4 \to \mathbb{Z}_4$ be a homomorphism such that $f(x) = \begin{cases} 0 & \text{if } x = 0, 2. \\ 2 & \text{if } x = 1, 3. \end{cases}$

Then $f(\mathbb{Z}_4) = \{0, 2\}$ is not a direct summand of $\mathbb{Z}_4$.

**Corollary 2.5.** Every completely reducible module is direct injective.

**Proof.** $M$ is completely reducible if and only if every submodule of $M$ is a direct summand of $M$ ([2]). It follows that a completely reducible module is direct injective.

**Corollary 2.6.** If $\text{End}(M)$ is (von Neumann) regular, then $M$ is direct injective.

**Lemma 2.7.** Let $0 \to L \to M \to N \to 0$ be a short exact sequence such that $L \oplus M$ is direct injective. Then this sequence splits.

**Proof.** Let $\nu_1 : L \to L \oplus M$, $\nu_2 : M \to L \oplus M$ be the corresponding canonical maps. By direct injectivity of $L \oplus M$, there exists a homomorphism $h \in \text{End}(L \oplus M)$ such that $\nu_1 = h \circ \nu_2 \circ g$. Define a homomorphism $f(m) = (\pi_L \circ h \circ \nu_2)(m)$, where $\pi_L : L \oplus M \to L$ is the corresponding projection map. Then $f \circ g = 1_L$ and hence the sequence splits.

**Proposition 2.8.** Let $\mu : K \to M$ be a monomorphism such that $M$ is injective. Then $K$ is injective if and only if $K \oplus M$ is direct injective.

**Proof.** Assume that $K \oplus M$ is direct injective, then we have a short exact sequence: $0 \to K \mu \to M \to \text{Im}(\mu) \to 0$. By Lemma 2.7, $M \cong K \oplus M / \text{Im}(\mu)$.

**Corollary 2.9.** The direct sum of two direct injective modules is direct injective if and only if every direct injective module is injective.

**Proof.** It is trivial because every module is isomorphic to a submodule of an injective module ([2]).

**Corollary 2.10.** A ring $R$ is completely reducible (=semisimple Artinian) if and only if every $R$-module is direct injective.

**Proof.** By Proposition 2.8 and Corollary 2.9, it is trivial.

**Definition 2.11.** A ring is called a left (right) dc-ring if every left (right) cyclic $R$-module is direct injective. A ring is called a dc-ring if it is a left and right dc-ring. Obviously every pc-ring is a dc-ring. A ring is said to be self
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Direct injective if $R$ is direct injective as an $R$-module. Trivially, any left $dc$-ring is self direct injective.

**Lemma 2.12.** Let $R$ be a ring and $I$ two sided ideal of $R$ contained in the annihilator of $M$. Then $M$ is direct injective over $R$ if and only if it is direct injective over $R/I$.

**Proposition 2.13.** A ring $R$ is left $dc$ if and only if $R/A$ is left $dc$ for each two sided ideal $A$ of $R$.

**Proof.** Let $R$ be a left $dc$-ring and $A$ an ideal of $R$. Let $I/A$ be any left ideal of $R/A$. Then, by [2], $(R/A)/(I/A)\cong R/I$ as an $R$-module. Since $A$ annihilates the $R$-module $R/I$, we may consider $R/I$ as $R/A$-module. Since $R$ is a left $dc$-ring, $R/I$ is $R$-direct injective. By Lemma 2.12, $R/I$, considered as an $R/A$-module, is $R/A$-direct injective. Hence any cyclic $R/A$-module is $R/A$-direct injective. i.e. $R/A$ is a duo-ring.

**Proposition 2.14.** Every factor ring of a $dc$-ring $R$ is self direct injective. Conversely if each factor ring of a $dc$-ring $R$ is self direct injective, then $R$ is a $dc$-ring.

**Proof.** Let $A$ be an ideal of a $duo$-ring $R$. Then $R/A$ is a $dc$-ring and hence self direct injective. Conversely, let $M$ be a cyclic $R$-module. Then $M\cong R/A$ for some left ideal $A$ of $R$. By hypothesis, $R/A$ is $R/A$-direct injective. Hence, by Lemma 2.12, $R/A$ is $R$-direct injective.

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**References**


