ON C.P-MODULES AND ANNIHILATOR SUBMODULES

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1. Introduction

As usual, a ring $R$ is called a left $p.p$-ring if every principal left ideal of $R$ is projective. In this paper, we define a left $R$-module $M$ is c.p-module if every cyclic submodule of $M$ is projective. R. Ware said $_RM$ is regular if $M$ is projective and every cyclic submodule of $M$ is a direct summand of $M$. Hence c.p-module is a generalization of $p.p$-ring and regular module.

Throughout $R$ will represent an associative ring with identity, and $R$-modules are unitary. For a subset $S$ of $_RM$, $I(S)=\{r\in R|rS=0\}$. Recall that the left singular submodule of $_RM$ is $Z(M)=\{x\in M|l(x)$ is large in $_RR\}$. $M$ is called singular (resp. non-singular) if $Z(M)=M$ (resp. $Z(M)=0$). For left $R$-module $M$, and any subset $T$ of $R$, define $r_M(T)=\{m\in M|Tm=0\}$. Note that $r_M(T)$ is not necessarily submodule of $M$. If any submodule $N$ of $M$ is the form $r_M(T)$ for some subset $T$ of $R$, $N$ is called annihilator submodule of $M$.

**LEMMA 1.** If $I$ is a right ideal of $R$, then $r_M(I)$ is a submodule of $M$ for any left $R$-module $M$.

**PROOF.** Let $r\in R$, $x\in r_M(I)$, then $l(rx)=(Ir)x\subseteq Ix=0$. Hence $rx\in r_M(I)$.

**LEMMA 2.** If $N$ is a submodule of $_RM$, then $r_M(l(N))$ is also submodule of $M$.

**PROOF.** Since $l(N)$ is an ideal [6, p.417, Theorem 1.4.], $r_M(l(N))$ is a submodule of $M$ by Lemma 1.

**LEMMA 3.** For any left $R$-module $M$, $M=r_M(l(M))$.

**PROOF.** If $M\neq r_M(l(M))$, there exists an $x$ in $M$ such that $x\in r_M(l(M))$. So $l(M)x\neq 0$ and since $l(M)\subseteq l(x)$, we have $l(x)x\neq 0$. But this is contradicts to $l(x)x=0$.

**LEMMA 4.** If $I$ is a large submodule of $_RM$, then $(I:x)_R=\{r\in R|rx\in I\}$ is
large in $\text{RR}$ for any $x$ of $M$.

PROOF. Let $K$ be a non-zero left ideal of $R$. If $Kx=0$, then $Kx \subseteq I$ and so $K \subseteq (I:x)_{\text{RR}}$. Hence $K \cap (I:x)_{\text{RR}} \neq 0$. If $Kx \neq 0$, then $Kx \cap I \neq 0$ since $I$ is large in $M$. Hence there exists a $kx(\neq 0)$ in $I$, where $k \in K$. Thus $k(\neq 0) \in (I:x)_{\text{RR}} \cap K$.

A submodule $A$ of a module $M$ is said to be a closed submodule of $M$ if $A$ has no proper large extensions inside $M$, that is, if $N$ is a submodule of $M$ and $A$ is large in $N$, then $A=N$.

**LEMMA 5.** $\text{RM}$ is non-singular if and only if $l(S)$ is closed in $\text{RR}$ for any subset $S$ of $M$.

PROOF. ($\Rightarrow$). Let $S$ be a subset of $M$ and $l(S)$ is large in $N$, where $N$ is a left ideal of $R$. For any $n \in N$, $(l(S):n)_{\text{RR}}$ is large in $\text{RR}$ by Lemma 4. Since $(l(S):n)_{\text{RR}}nS=0$, $nSCZ(M)$. By hypothesis, $nS=0$ and so $n \in l(S)$. Therefore $N=l(S)$.

($\Leftarrow$). If $x \in Z(M)$, then there exists a large left ideal $L$ such that $Lx=0$. Since $L$ is contained in $l(x)$, $l(x)$ is large in $\text{RR}$. But $l(x)$ is closed in $\text{RR}$ by hypothesis. So $l(x)=R$ and we have $x=0$.

2. c. p-modules

A ring $A(\neq 0)$ is called a left (resp. right) $s$-unital ring if $a \in Aa$ (resp. $a \in aA$).

**LEMMA 6.** If $F$ is a finite subset of a right $s$-unital ring (resp. an $s$-unital ring) $A$, then there exists an element $e$ in $A$ such that $ae=a$ (resp. $ea=ae=a$) for all $a$ of $F$.

PROOF. [1, Theorem 1].

**THEOREM 1.** The following statements are equivalent:

(1) $Rm$ is projective left $R$-module for any $m$ of $\text{RM}$.

(2) $l(m)$ is a direct summand of $\text{RR}$ for any $m$ of $\text{RM}$.

(3) $Rm$ is isomorphic to a direct summand of $\text{RR}$ for any $m$ of $\text{RM}$.

(4) $Rm$ is flat and $l(m)$ is finitely generated left ideal of $R$ for any $m$ of $\text{RM}$.

PROOF. (1) $\Rightarrow$ (2). Since $Rm \cong R/l(m)$, it is obvious.

(2) $\Rightarrow$ (3). $l(m)=l(e)$, $e=e^2 \in R$ by [10, Theorem 2]. Hence $Rm \cong R/l(m)$ =
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\( R/l(e) \cong Re. \)

(3) \( \implies \) (1). \( _RM \) is projective if and only if it is isomorphic to a direct summand of a free module \([4, \text{p.} 84, \text{Corollary}]. \) Hence \( Rm \) is projective for any \( m \) of \( _RM \) since \( _RR \) is free.

(4) \( \implies \) (2). \( R/l(m) \cong Rm \) is a flat left \( R \)-module if and only if \( l(m) \) is a right \( s \)-unital ring \([1, \text{Proposition 1}]. \) Let \( l(m) = Ra_1 + Ra_2 + \cdots + Ra_n (a_i \in R), \) then by Lemma 6 there exists an element \( e \) in \( l(m) \) such that \( a_i e = a_i \) for all \( i = 1, 2, \cdots n. \) That is \( l(m) \) has right identity, hence \( l(m) \) is a direct summand of \( _RR \) by \([10, \text{Theorem 2}]. \)

(2) \( \implies \) (4). Since every projective module is flat, it is obvious.

We call \( _RM \) satisfying the equivalent conditions of Theorem 1 a \( c.p \)-module. Examples of \( c.p \)-modules.

(1) Left \( p.p \)-ring is a special case of \( c.p \)-module since every \( c.p \)-module \( _RM \) is a left \( p.p \)-ring when \( R = M. \)

(2) Every regular module is \( c.p \)-module \([7, \text{proposition 2.1}]. \)

**THEOREM 2.** Let \( _RM \) is \( c.p \)-module, then

(1) \( M \) is non-singular.

(2) \( \text{Soc } M \) is projective.

(3) Every submodule of \( _RM \) is \( c.p \)-module.

**PROOF.** (1). Let \( m(\neq 0) \) is an element of \( M, \) then there exists a left ideal \( J \) of \( R \) such that \( l(m) \oplus J = R. \) Since \( l(m) \neq R, \) \( J \neq 0. \) So \( l(m) \) is not large left ideal of \( R, \) that is \( m \notin Z(M). \)

(2). Since \( \text{soc } M \) is a direct sum of simple submodules of \( M, \) every simple module is cyclic, we have \( \text{soc } M \) is a direct sum of cyclic submodules of \( M. \) By hypothesis and \([4, \text{p.} 82, \text{Proposition 3}]. \) \( \text{soc } M \) is projective.

(3). Trivial.

**COROLLARY 1.** In left \( p.p \)-ring \( R, \) \( \text{soc}(R) \) is a direct summand of \( R_R \) if and only if \( \text{soc}(R) \) is finitely generated right ideal of \( R. \)

**PROOF.** Since \( R \) is left non-singular ring, \( (R/\text{soc}(R))_R \) is flat \([5, \text{p.} 37, \text{Exercises 24}]. \) Hence \( \text{soc}(R) \) is a left \( s \)-unital ring \([1, \text{Proposition 1}]. \) So \( \text{soc}(R) \) is a direct summand of \( R_R \) if and only if \( \text{soc}(R) \) is finitely generated right ideal of \( R. \)
THEOREM 3. \( \{R_{M_i}\}_{i \in I} \) is c.p-module for each \( i \in I \) if and only if \( \sum_{i \in I} M_i \) is c.p-module.

PROOF. \((\Longrightarrow)\). Let \( x \in \sum_{i \in I} M_i \), then \( x = x_{i_1} + x_{i_2} + \cdots + x_{i_n} \), \( x_{i_j} \in M_{i_j} \), \( j = 1, 2, \ldots, n \). Then \( Rx = Rx_{i_1} \oplus Rx_{i_2} \oplus \cdots \oplus Rx_{i_n} \). Since each \( Rx_{i_j} \) is projective, \( Rx \) is also projective.

\((\Longleftarrow)\). Trivial.

LEMMA 7. If \( R \) is a left p.p-ring, then \( \text{soc}(R) \) is an idempotent ideal.

PROOF. Since the class of all non-singular left \( R \)-modules is closed under submodule [5, p.32, Proposition 1.22(a)], \( Z(\text{soc}(R)) = 0 \). Hence \( \text{soc}(\text{soc}(R)) = \text{soc}(R) \text{soc}(R) \) [5, p.35, Corollary 1.23]. So that we have \( \text{soc}(R) = (\text{soc}(R))^2 \).

LEMMA 8. If the Jacobson radical \( J(R) \) of a ring \( R \) is projective, then \( R \) is semiprimitive.

PROOF. If \( J(R) \) is non-zero, then \( J(J(R)) = J(R)J(R) \). But this contradicts to \( J(R)J(R) \neq J(R) \).

LEMMA 9. If \( _RM \) is c.p-module, then the intersection of all maximal submodules of \( \text{soc}(R)M \) (denoted by \( J(\text{soc}(R)M) \)) is \( J(R)\text{soc}M \).

PROOF. Since \( _RM \) is non-singular, \( \text{soc}M = \text{soc}(R)M \), and \( J(R)\text{soc}M = J(\text{soc}M) \) from the fact that \( \text{soc}M \) is projective. Hence we have \( J(\text{soc}(R)M) = J(\text{soc}M) = J(R)\text{soc}M \).

3. Annihilator submodules

A left \( R \)-module \( M \) is faithful if \( I(m) = 0 \).

THEOREM 4. If \( N \) is a maximal submodule of \( _RM \), then \( N \) is either annihilator submodule of \( M \) or \( I(N) = I(M) \), but not both.

PROOF. Since \( I(N)N = 0 \), \( N \subseteq r_M(I(N)) \) By maximality of \( N \), \( N = r_M(I(N)) \) or \( r_M(I(N)) = M \) but not both. If \( r_M(I(N)) = M \), then \( I(N)M = 0 \). So we have \( I(N) \subseteq I(M) \). But \( I(M) \subseteq I(N) \) since \( N \subseteq M \), so that \( I(N) = I(M) \). Next, if \( N = r_M(S) \) for some subset \( S \) of \( R \), then \( SN = 0 \) and so \( N = r_M(S) \supseteq r_M(I(N)) \). Hence \( r_M(I(N)) \)...
\begin{align*}
\neq M. \text{ If } l(N) = l(M), \text{ then } M = r_M(l(M)) = r_M(l(N)) \neq M, \text{ a contradiction.} \\
\end{align*}

**COROLLARY 2.** If \( M \) is a maximal left ideal of \( R \), then \( M \) is either two sided ideal of \( R \) or \( M \) is faithful as a left \( R \)-module.

**PROOF.** Either \( M = r_M(l(M)) \) or \( l(M) = l(N) \). Since \( l(M) \) is an ideal and \( l(R) = 0 \), \( M \) is two-sided or \( l(M) = 0 \).

**THEOREM 5.** The following statements are equivalent:

1. \( R \) is a regular ring.
2. Every cyclic \( R \)-module is \( p \)-injective.
3. Every semisimple \( R \)-module is \( p \)-injective.

**PROOF.** (1) \( \iff \) (2), [2, Theorem 2]

3. \( \implies \) (2). Let \( M \) be a cyclic \( R \)-module, then \( M \) is simple and so semisimple. Hence \( M \) is \( p \)-injective.

(1) \( \implies \) (3). Trival.

**THEOREM 6.** The following statements are equivalent:

1. \( R \) is completely reducible.
2. \( R \) is left non-singular and every large left ideal is left annihilator.
3. \( R \) is a semi-prime ring whose large left ideals are left annihilator.
4. \( R \) is a semi-prime ring whose maximal left ideals are left annihilator.
5. \( R \) is a left \( V \)-ring whose maximal left ideals are left annihilator.
6. \( R \) is a fully left idempotent ring whose maximal left ideals are left annihilator.

7. \( R \) is a right \( V \)-ring whose maximal left ideals are left annihilator.
8. \( R \) is a right \( p \)-\( V \)-ring whose maximal left ideals are left annihilator.
9. \( R \) is a right \( p \)-\( V \)-ring whose large left ideals are left annihilator.

10. Every cyclic \( R \)-module is projective.
11. Every \( R \)-module is non-singular.
12. Every simple \( R \)-module is non-singular.
13. Every semisimple \( R \)-module is projective.
14. Every semisimple \( R \)-module is injective.

(2) \( \iff \) (9). The right-left analogues of (2)\( \iff \) (9).

**PROOF.** The implications (1) \( \implies \) (7) \( \implies \) (8) and (11) \( \implies \) (12) are obvious. (8) \( \implies \) (1). Every right \( p \)-\( V \)-ring is fully right idempotent and every fully right idempotent ring is a left non-singular [1, Proposition 6 and 7]. So every max-
imal left ideal is not large since every annihilator is closed in \( _RR \) by Lemma 5. Thus \( R \) is completely reducible.

(9) \( \Rightarrow \) (1). Since \( R \) is left non-singular, \( R \) has no proper large left ideal. Therefore \( R \) is completely reducible.

(10) \( \Rightarrow \) (1). For any maximal left ideal \( M \) of \( R \), \( R M \) is a simple \( R \)-module. Since every simple \( R \)-module is cyclic, \( R/M \) is projective and so \( M \) is a direct summand of \( R \).

(12) \( \Rightarrow \) (1). Since every simple \( R \)-module is either singular or projective [5, Proposition 1. 24], \( R/M \) is projective for any maximal left ideal \( M \) of \( R \). Hence \( R \) is completely reducible.

(13) \( \Rightarrow \) (1). Let \( A \) be an any simple \( R \)-module, then \( A \) is semisimple and so projective. Hence every simple \( R \)-module is projective.

(14) \( \Rightarrow \) (1). [9, Theorem 3.2].

(1) \( \Rightarrow \) (9), (1) \( \Rightarrow \) (10), (1) \( \Rightarrow \) (13), (1) \( \Rightarrow \) (14) are trivial.

(1) \( \Rightarrow \) (11). If \( R \) is completely reducible, every \( R \)-module is completely reducible and so every \( R \)-module is a c.p-module. Hence every \( R \)-module is non-singular from Theorem 2.

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REFERENCES