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ON $C.P$-MODULES AND ANNIHILATOR SUBMODULES

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1. Introduction

As usual, a ring R is called a *left p.p-ring* if every principal left ideal of R is projective. In this paper, we define a left R-module M is *c.p*-module if every cyclic submodule of M is projective. R. Ware said ${}_RM$ is regular if M is projective and every cyclic submodule of M is a direct summand of M. Hence *c.p*-module is a generalization of *p.p*-ring and regular module.

Throughout R will represent an associative ring with identity, and R-modules are unitary. For a subset S of ${}_RM$, $l(S)=\{r\in R\,|\,rS=0\}$. Recall that the left singular submodule of ${}_RM$ is $Z(M)=\{z\in M\,|\,l(z)$ is large in ${}_RR\}$. M is called *singular* (resp. *non-singular*) if $Z(M)=M$(resp. $Z(M)=0$). For left R-module M, and any subset T of R, define $r_M(T)=\{m\in M\,|\,Tm=0\}$. Note that $r_M(T)$ is not necessarily submodule of M. If any submodule N of M is the form $r_M(T)$ for some subset T of R, N is called *annihilator submodule* of M.

LEMMA 1. *If I is a right ideal of R, then $r_M(I)$ is a submodule of M for any left R-module M.*

PROOF. Let $r\in R$, $x\in r_M(I)$, then $I(rx)=(Ir)x\subset Ix=0$. Hence $rx\in r_M(I)$.

LEMMA 2. *If N is a submodule of ${}_RM$, then $r_M(l(N))$ is also submodule of M.*

PROOF. Since $l(N)$ is an ideal [6, p.417, Theorem 1.4.], $r_M(l(N))$ is a submodule of M by Lemma 1.

LEMMA 3. For any left R-module M, $M=r_M(l(M))$.

PROOF. If $M\neq r_M(l(M))$, there exists an x in M such that $x\notin r_M(l(M))$. So $l(M)x\neq 0$ and since $l(M)\subset l(x)$, we have $l(x)x\neq 0$. But this is contradicts to $l(x)x=0$.

LEMMA 4. *If I is a large submodule of ${}_RM$, then $(I:x)_R=\{r\in R\,|\,rx\in I\}$ is* *large in $_RR$ for any x of M.*

PROOF. Let K be a non-zero left ideal of R. If $Kx=0$, then $Kx\subset I$ and so $K\subset(I:x)_R$. Hence $K\cap(I:x)_R\neq 0$. If $Kx\neq 0$, then $Kx\cap I\neq 0$ since I is large in M. Hence there exists a $kx(\neq 0)$ in I, where $k\in K$. Thus $k(\neq 0)\in(I:x)_R\cap K$.

A submodule A of a module M is said to be a *closed submodule* of M if A has no proper large extensions inside M, that is, if N is a submodule of M and A is large in N, then $A=N$.

LEMMA 5. *$_RM$ is non-singular if and only if $l(S)$ is closed in $_RR$ for any subset S of M.*

PROOF. ($\Longrightarrow$). Let S is a subset of M and $l(S)$ is large in N, where N is a left ideal of R. For any $n\in N$, $(l(S):n)_R$ is large in $_RR$ by Lemma 4. Since $(l(S):n)_R nS=0$, $nS\subset Z(M)$. By hypothesis, $nS=0$ and so $n\in l(S)$. Therefore $N=l(S)$.

($\Longleftarrow$). If $x\in Z(M)$, then there exists a large left ideal L such that $Lx=0$. Since L is contained in $l(x)$, $l(x)$ is large in $_RR$. But $l(x)$ is closed in $_RR$ by hypothesis. So $l(x)=R$ and we have $x=0$.

2. c. p-modules

A ring $A(\neq 0)$ is called a *left*(resp. *right*) *s-unital* ring if $a\in Aa$(resp. $a\in aA$).

LEMMA 6. *If F is a finite subset of a right s-unital ring (resp. an s-unital ring) A, then there exists an element e in A such that $ae=a$(resp. $ea=ae=a$) for all a of F.*

PROOF. [1, Theorem 1].

THEOREM 1. *The following statements are equivalent:*

(1) *Rm is projective left R-module for any m of $_RM$.*

(2) *$l(m)$ is a direct summand of $_RR$ for any m of $_RM$.*

(3) *Rm is isomorphic to a direct summand of $_RR$ for any m of $_RM$.*

(4) *Rm is flat and $l(m)$ is finitely generated left ideal of R for any m of $_RM$.*

PROOF. (1) $\Longrightarrow$ (2). Since $Rm\cong R/l(m)$, it is obvious.

(2) $\Longrightarrow$ (3). $l(m)=l(e)$, $e=e^2\in R$ by [10, Theorem 2]. Hence $Rm\cong R/l(m)=$

$R/l(e)\cong Re$.

(3) $\Longrightarrow$ (1). ${}_RM$ is projective if and only if it is isomorphic to a direct summand of a free module [4, p. 84, Corllary]. Hence Rm is projective for any m of ${}_RM$ since ${}_RR$ is free.

(4) $\Longrightarrow$ (2). $R/l(m)\cong Rm$ is a flat left R-module if and only if $l(m)$ is a right s-unital ring [1, Proposition 1]. Let $l(m)=Ra_1+Ra_2+\cdots+Ra_n(a_i\in R)$, then by Lemma 6 there exists an element e in $l(m)$ such that $a_ie=a_i$ for all $i=1, 2, \cdots n$. That is $l(m)$ has right identity, hence $l(m)$ is a direct summand of ${}_RR$ by [10, Theorem 2].

(2) $\Longrightarrow$ (4). Since every projective module is flat, it is obvious.

We call ${}_RM$ satisfying the equivalent conditions of Theorem 1 *c.p-module.*

Examples of *c.p*-modules.

(1) Left *p.p*-ring is a special case of *c.p*-module since every *c.p.* module ${}_RM$ is a left *p.p*-ring when $R=M$.

(2) Every regular module is *c.p*-module [7, proposition 2.1].

THEOREM 2. *Let* ${}_RM$ *is c.p-module, then*

(1) *M is non-singular.*

(2) *Soc M is projective.*

(3) *Every submodule of* ${}_RM$ *is c.p-module.*

PROOF. (1). Let $m(\neq 0)$ is an element of M, then there exists a left ideal J of R such that $l(m)\oplus J=R$. Since $l(m)\neq R$, $J\neq 0$. So $l(m)$ is not large left ideal of R, that is $m\notin Z(M)$.

(2). Since soc M is a direct sum of simple submodules of M, every simple module is cyclic, we have soc M is a direct sum of cyclic submodules of M. By hypothesis and [4, p. 82, Proposition 3], soc M is projective.

(3). Trivial.

COROLLARY 1. *In left p.p-ring R,* $soc({}_RR)$ *is a direct summand of* R_R *if and only if* $soc({}_RR)$ *is finitely generated right ideal of R.*

PROOF. Since R is left non-singular ring, $(R/\mathrm{soc}({}_RR))_R$ is flat [5, p. 37, Exercises 24]. Hence $\mathrm{soc}({}_RR)$ is a left s-unital ring [1, Proposition 1]. So $\mathrm{soc}({}_RR)$ is a direct summand of R_R if and only if $\mathrm{soc}({}_RR)$ is finitely generated right ideal of R.

THEOREM 3. *$\{_RM_i\}_{i\in I}$ is c.p-module* for each *$i\in I$ if and only if $\sum_{i\in I} M_i$ is c.p-module.*

PROOF. ($\Longrightarrow$). Let $x\in \sum_{i\in I} M_i$, then $x=x_{i_1}+x_{i_2}+\cdots+x_{i_n}$, $x_{i_j}\in M_{i_j}$, $j=1, 2, \cdots, n$. Then $Rx=Rx_{i_1}\oplus Rx_{i_2}\oplus\cdots\oplus Rx_{i_n}$. Since each Rx_{i_j} is projective, Rx is also projective.

($\Longleftarrow$). Trivial.

LEMMA 7. *If R is a left p.p-ring, then $soc(_RR)$ is an idempotent ideal.*

PROOF. Since the class of all non-singular left R-modules is closed under submodule [5, p. 32, Proposition 1.22(a)], $Z(\text{soc}(_RR))=0$. Hence $\text{soc}(\text{soc}(_RR))=\text{soc}(_RR)\text{soc}(_RR)$ [5, p. 35, Corollary 1.26]. So that we have $\text{soc}(_RR)=(\text{soc}(_RR))^2$.

LEMMA 8. *If the Jacobson radical $J(R)$ of a ring R is projective, then R is semiprimitive.*

PROOF. If $J(R)$ is non-zero, then $J(J(R))=J(R)J(R)$. But this contradicts to $J(R)J(R)\neq J(R)$.

LEMMA 9. *If $_RM$ is c.p-module, then the intersection of all maximal submodules of $soc(_RR)M$ (denoted by $J(soc(_RR)M)$ is $J(R)socM$.*

PROOF. Since $_RM$ is non-singular, $\text{soc}M=\text{soc}(_RR)M$, and $J(R)\text{soc}M=J(\text{soc}M)$ from the fact that $\text{soc}M$ is projective. Hence we have $J(\text{soc}(_RR)M)=J(\text{soc}M)=J(R)\text{soc}M$.

3. Annihilator submodules

A left R-module M is *faithful* if $l(m)=0$.

THEOREM 4. *If N is a maximal submodule of $_RM$, then N is either annihilator submodule of M or $l(N)=l(M)$, but not both.*

PROOF. Since $l(N)N=0$, $N\subset r_M(l(N))$ By maximality of N, $N=r_M(l(N))$ or $r_M(l(N))=M$ but not both. If $r_M(l(N))=M$, then $l(N)M=0$. So we have $l(N)\subset l(M)$. But $l(M)\subset l(N)$ since $N\subset M$, so that $l(N)=l(M)$. Next, if $N=r_M(S)$ for some subset S of R, then $SN=0$ and so $N=r_M(S)\supset r_M(l(N))$. Hence $r_M(l(N))$

$\neq M$. If $l(N)=l(M)$, then $M=r_M(l(M))=r_M(l(N))\neq M$, a contradiction.

COROLLARY 2. *If M is a maximal left ideal of R, then M is either two sided ideal of R or M is faithul as a left R-module.*

PROOF. Either $M=r_M(l(M))$ or $l(M)=l(N)$. Since $l(M)$ is an ideal and $l(R)=0$, M is two-sided or $l(M)=0$.

THEOREM 5. *The following statements are equivalent:*

(1) *R is a regular ring.*

(2) *Every cyclic R-module is p-injective.*

(3) *Every semisimple R-module is p-injective.*

PROOF. (1) $\Longleftrightarrow$ (2). [2, Theorem 2]

(3) $\Longrightarrow$ (2). Let M be a cyclic R-module, then M is simple and so semisimple. Hence M is p-injective.

(1) $\Longrightarrow$ (3). Trival.

THEOREM 6. *The following statments are equivalent:*

(1) *R is completely reducible.*

(2) *R is left non-singular and every large left ideal is left annihilator.*

(3) *R is a semi-prime ring whose large left ideals are left annihilator.*

(4) *R is a semi-prime ring whose maximal left ideals are left annihilator.*

(5) *R is a left V-ring whose maximal left ideals are left annihilator.*

(6) *R is a fully left idempotent ring whose maximal left ideals are left annihilator.*

(7) *R is a right V-ring whose maximal left ideals are left annihilator.*

(8) *R is a right p-V-ring whose maximal left ideals are left annihilator.*

(9) *R is a right p-V-ring whose large left ideals are left annihilator.*

(10) *Every cyclic R-module is projective.*

(11) *Every R-module is non-singular.*

(12) *Every simple R-module is non-singular.*

(13) *Every semisimple R-module is projective.*

(14) *Every semisimple R-module is injective.*

(2)′—(9)′ The right-left analogues of (2)—(9).

PROOF. The implications (1)$\Longrightarrow$(7)$\Longrightarrow$(8) and (11)$\Longrightarrow$(12) are obvious. (8) $\Longrightarrow$(1). Every right p-V-ring is fully right idempotent and every fully right idempotent ring is a left non-singular [1, Proposition 6 and 7]. So every maximal left ideal is not large since every annihilator is closed in $_RR$ by Lemma 5. Thus R is completely reducible.

(9)$\Longrightarrow$(1). Since R is left non-singular, R has no proper large left ideal. Therefore R is completely reducible.

(10)$\Longrightarrow$(1). For any maximal left ideal M of R, R/M is a simple R-module. Since every simple R-module is cyclic, R/M is projective and so M is a direct summand of R.

(12)$\Longrightarrow$(1). Since every simple R-module is either singular or projective [5, Proposition 1. 24], R/M is projective for any maximal left ideal M of R. Hence R is completely reducible.

(13)$\Longrightarrow$(1). Let A be an any simple R-module, then A is semisimple and so projective. Hence every simple R-module is projective.

(14)$\Longrightarrow$(1). [9, Theorem 3.2].

(1)$\Longrightarrow$(9), (1)$\Longrightarrow$(10), (1)$\Longrightarrow$(13), (1)$\Longrightarrow$(14) are trivial.

(1)$\Longrightarrow$(11). If R is completely reducible, every R-module is completely reducible and so every R-module is a *c.p*-module. Hence every R-module is non-singular from Theorem 2.

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