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COMPLETE LIFT OF F-STRUCTURE MANIFOLD

By Lovejoy S.K. Das

1. Introduction

Let F be a non zero tensor field of type (1,1) and of class C^∞ on an n-dimensional manifold V_n such that [1]

(1.1) $$F^K+(-)^{K+1}F=0 \text{ and } F^W+(-)^{W+1}F\neq 0 \text{ for } 1<W<K$$

where K is a fixed positive integer greater than 2. Such a structure on V_n is called an *F-strcuture of rank 'r' and degree K.* If the rank of F is a constant and $r=r(F)$, then V_n is called an *F-structure manifold of degree* $K(\geq 3)$. The case when K is odd has been considered in this paper.

Let the operators on V_n be defined as follows [1]

(1.2) $$l=(-)^K F^{K-1} \text{ and } m=I+(-)^{K+1}F^{K-1}$$

where I denotes the identity operator on V_n.

From the operators defined by (1.2) we have [1]

(1.3) $$l+m=I \text{ and } l^2=l, \text{ and } m^2=m.$$

For F-satisfying (1.1), there exist complementary distributions L and M corresponding to the projection operators l and m respectively.

If rank (F)=constant on V_n then dim $L=r$, and dim $M=(n-r)$. We have following results [1]

(1.4) (a) $Fl=lF=F$ and $Fm=mF=0$
(b) $F^{K-1}l=-l$ and $F^{K-1}m=0$

2. Complete lift of F-structure in tangent bundle

Let V_n be an n-dimensional differentiable manifold of class C^∞ and $T_p(V_n)$ the tangent space at a point P of V_n and $T(V_n)=\bigcup_{P\in V_n} T_P(V_n)$ is the tangent bundle over the manifold V_n.

Let us denote by $\mathscr{T}^r_s(V_n)$, the set of all tensor fields of class C^∞ and of type (r,s) in V_n and $T(V_n)$ be the tangent bundle over V_n. The complete lifts F^C of an element of $\mathscr{T}^1_1(V_n)$ with local components F^h_i has components of the form [2]

(2.1) $$F^C : \begin{pmatrix} F_i^h & 0 \\ \partial F_i^h & F_h^i \end{pmatrix}$$

Now we obtain the following results on the complete lift of F satisfying (1.1).

THEOREM 2.1. *For $F \in \mathcal{T}_1^1(V_n)$, the complete lift F^C of F is an F structure if it is for F also. Then F is of rank r, iff F^C is of rank $2r$.*

PROOF. Let F, $G \in \mathcal{T}_1^1(V_n)$. Then we have [2]

(2.2) $$(FG)^C = F^C G^C$$

Replacing G by F in (2.2) we obtain

$$(FF)^C = F^C F^C$$

(2.3) $$\text{or, } (F^2)^C = (F^C)^2$$

Now putting $G = F^{K-1}$ in (2.2) since G is (1,1) tensor field therefore F^{K-1} is also (1,1) so we obtain

$(F\ F^{K-1})^C = F^C (F^{K-1})^C$ which in view of (2.3) becomes

(2.4) $$(F^K)^C = (F^C)^K$$

Taking complete lift on both sides of equation (1.1) we get

$$(F^K)^C + ((-)^{K+1} F)^C = 0$$

which is in consequence of equation (2.4) gives

(2.5) $$(F^C)^K + (-)^{K+1} F^C = 0$$

Thus equation (1.1) and (2.5) are equivalent. The second part of the theorem follows in view of equation (2.1). Let F satisfying (1.1) be an F-structure of rank r in V_n. Then the complete lifts l^C of l and m^C of m are complementary projection tensors in $T(V_n)$. Thus there exist in $T(V_n)$ two complementary distributions L^C and M^C determined by l^C and m^C respectively.

3. Integrability conditions of F-structure in tangent bundle

Let $F \in \mathcal{T}_1^1(V_n)$, then the Nijenhuis tensor N_F of F satisfying (1.1) is a tensor field of the type (1.2) given by [2]

(3.1) a) $N_F(X,Y) = [FX, FY] - F[FX, Y] - F[X, FY] + F^2[X, Y]$.

Let N^C be the Nijenhuis tensor of F^C in $T(V_n)$ of F in V_n, then we have

b) $N^C(X^C, Y^C)=[F^CX^C, F^CY^C]-F^C[F^CX^C, Y^C]$
$-F^C[X^C, F^CY^C]+(F^2)^C[X^C, Y^C].$

For any X, $Y\in\mathscr{T}^1_0(V_n)$ and $F\in\mathscr{T}^1_1(V_n)$ we have [2]

(3.2) a) $[X^C, Y^C]=[X, Y]^C$ and $(X+Y)^C=X^C+Y^C$

b) $F^CX^C=(FX)^C$.

From (1.4)a and (3.2)b we have

(3.3) $$F^Cm^C=(Fm)^C=0.$$

THEOREM 3.1. *The following identities hold*

(3.4) (i) $$N^C(m^CX^C, m^CY^C)=(F^C)^2[m^CX^C, m^CY^C],$$

(3.5) (ii) $$m^CN^C(X^C, Y^C)=m^C[F^CX^C, F^CY^C],$$

(3.6) (iii) $$m^CN^C(l^CX^C, l^CY^C)=m^C[F^CX^C, F^CY^C],$$

(3.7) (iv) $$(-)^Km^CN^C((F^C)^{K-2}X^C, (F^C)^{K-2}Y^C)=$$
$$=m^C[l^CX^C, l^CY^C].$$

PROOF. The proofs of (3.4) to (3.7) follow by virtue of equations (1.4), (3.1)b and (3.3).

THEOREM 3.2. *For any X, $Y\in\mathscr{T}^1_0(V_n)$, the following conditions are equivalent.*

(i) $m^CN^C(X^C, Y^C)=0$,

(ii) $m^CN^C(l^CX^C, l^CY^C)=0$,

(iii) $(-)^Km^CN^C((F^{K-2})^CX^C, (F^{K-2})^CY^C)=0$.

PROOF. In consequence of equation (3.1)b and equation (1.4), it can be easily proved that $N^C(l^CX^C, l^CY^C)=0$ iff $(-)^KN^C((F^{K-2})^CX^C, (F^{K-2})^CY^C)=0$ for all X, $Y\in\mathscr{T}^1_0(V_n)$.

Now r.h.s. of equations (3.5) and (3.6) are equal which in view of the above equation shows that conditions (i), (ii) & (iii) are equivalent to each other.

THEOREM 3.3 *The complete lift M^C of the distribution M in $T(V_n)$ is integrable iff M is integrable in V_n.*

PROOF. It is known that the distribution M is integrable in V_n iff [3].

(3.8) $$l[mX,\ mY]=0 \text{ for any } X,\ Y\in\mathscr{T}_0^1(V_n)$$

Taking complete lift of both sides of (3.8) we get

(3.9) $$l^C[m^C X^C,\ m^C Y^C]=0$$

where $l^C=(I-m)^C=I-m^C$, is the projection tensor complementary to m^C. Thus the conditions (3.8) and (3.9) are equivalent.

THEOREM 3.4. *For any* $X,\ Y\in\mathscr{T}_0^1(V_n)$, *let the distribution* M *be integrable in* V_n *iff* $N(mX,\ mY)=0$. *Then the distribution* M^C *is integrable in* $T(V_n)$ *iff*

$$l^C N^C(m^C X^C,\ m^C Y^C)=0 \text{ or equivalently,}$$
$$N^C(m^C X^C,\ m^C Y^C)=0.$$

PROOF. By virtue of condition (3.4) we have

$$N^C(m^C X^C,\ m^C Y^C)=(F^C)^2[m^C X^C,\ m^C Y^C].$$

Multiplying throughout by l^C we get

$$l^C N^C(m^C X^C,\ m^C Y^C)=(E^C)^2 l^C[m^C X^C,\ m^C Y^C]$$

which in view of equation (3.9) becomes

(3.10) $l^C N^C(m^C X^C,\ m^C Y^C)=0$. Also in view of (3.3) we have

(3.11) $m^C N^C(m^C X^C,\ m^C Y^C)=0$.

Adding (3.10) and (3.11), we obtain $(l^C+m^C)N^C(m^C X^C,\ m^C Y^C)=0$.
Since $l^C+m^C=I^C=I$, we have $N^C(m^C X^C,\ m^C Y^C)=0$.

THEOREM 3.5 *For any* $X,\ Y\in\mathscr{T}_0^1(V_n)$ *let the distribution* L *be integrable in* V_n *that is* $mN(X,\ Y)=0$ *then the distribution* L^C *in integrable in* $T(V_n)$ *iff anyone of the conditions of theorem* (3.2) *is satisfied.*

PROOF. The distribution L is integrable in V_n iff $m[lX,\ lY]=0$.

Thus distribution L^C is integrable in $T(V_n)$ iff $m^C[l^C X^C,\ l^C Y^C]=0$.
so the theorem follow by making use of equation (3.7). We now define following

(i) distribution L is integrable

(ii) an arbitrary vector field Z tangent to an integral manifold of L.

(iii) the operator $\overset{*}{F}$, such that $\overset{*}{F}\ Z=FZ$.

In view of equation (1.4), the induced structure $\overset{*}{F}$ of F is an almost complex structure on each integral manifold of L and $\overset{*}{F}$ makes tangent spaces invariant of every integral manifold of L.

DEFINITION. We say that F-structure is ***partially integrable*** if the distribution L is integrable and the almost complex structure F induced from $\overset{*}{F}$ on each integral manifold of L is also integrable.

Let us denote the vector valued 2-form $\overset{*}{N}(Z,W)$, the Nijenhuis tensor corresponding to the Nijenhuis tensor of the almost complex structure inuced from F-structure on each integral manifold of L and for any two $Z,W\in\mathscr{T}_0^1(V_n)$ tangent to an integral manifold of L, then we have

$$(3.12)\qquad \overset{*}{N}(Z,W)=[\overset{*}{F}Z,\overset{*}{W}]-\overset{*}{F}[\overset{*}{F}Z,W]-\overset{*}{F}[Z,\overset{*}{F}W]+\overset{*2}{F}[Z,W]$$

which in view of (3.1)b and (3.12) yields

$$(3.13)\qquad N^C(l^CX^C,\ l^CY^C)=\overset{*}{N}{}^C(l^CX^C,\ l^CY^C)$$

THEOREM 3.6. *For any X, $Y\in\mathscr{T}_0^1(V_n)$ let the F-structure be partially integrable in V_n i.e., $N(lX,\ lY)=0$. Then the necessary and sufficient conditon for F-structure to be partially integrable in $T(V_n)$ is that $N^C(l^CX^C,\ l^CY^C)=0$ or equivalently*

$$N^C((F^{K-2})^CX^C,\ (F^{K-2})^CY^C)=0$$

PROOF. In view of equation (1.4) and equation (3.1)b, we can prove easily that

$$N^C(l^CX^C,\ l^CY^C)=0 \text{ iff } N^C((F^{K-2})^CX^C,\ (F^{K-2})^CY^C)=0$$

for any X, $Y\in\mathscr{T}_0^1(V_n)$.

Now by making use of (3.13) and theorem (3.5) the result follows immediately.

When both distributions L and M are integrable we can choose a local coordinate system such that all L and M are represented by putting $(n-r)$ local coordinate constant and r-coordinate constant respectively. We call such a coordinate system an ***adapted coordinate system.*** It can be supposed that in an adapted coordinate system the projection operators l and m have the component of the form

$$l=\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, \qquad m=\begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix}$$

respectively where I_r denotes the unit matrix of order 'r' and I_{n-r} is of order $(n-r)$.

Since F satisfies equation (1.4)a, the tensor F has components of the form

$$F=\begin{pmatrix} F_r & 0 \\ 0 & 0 \end{pmatrix}$$

is an adapted coordinate system where F_r denotes $r\times r$ square matrix.

DEFINITION We say that an F-structure is *integrable* if

(i) The structure F is partially integrable,

(ii) The distribution M is integrable i.e. $N(mX, mY)=0$

(iii) The components of the F-structure are independent of the coordinates which are constant along the integral manifold of L in an adapted system.

THEOREM 3.7 *For any* $X, Y\in\mathscr{T}^1_0(V_n)$ *let* F *structure to be integrable in* V_n *iff* $N(X, Y)=0$. *Then the* F-*structure is integrable in* $T(V_n)$ *iff*

$$N^C(X^C, Y^C)=0.$$

PROOF. In view of equations (3.1)a and (3.1)b we get

$$N^C(X^C, Y^C)=(N(X, Y))^C,$$

since F-structure is integrable in V_n thus we obtain the result.

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Department of Mathematics
and Astronomy,
Lucknow Universiy,
Lucknow (INDIA)

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