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〈Original〉

Stress Distribution in an Infinite Plate Containing an Elliptical Crack (Part I)

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楕圓形 크랙을 포함하는 無限平板의 應力解析(第一報)

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抄 錄

이 論文에서는 楕圓形의 크랙을 包含하는 有限한 두께를 가진 isotropic彈性體의 三次元應力解析을 다루었다. 크랙은 平板의 面에 나란하고 그 中立面에 位置하며 일정한 引張力이 平板의 面에 作用하고 있다.

問題를 解析하기 위하여 二重 Fourier 積分變換을 使用하여 應力解析이 제 일종 Fredholm 적분 방정식의 解로 될 수 있음을 보였다. 두 극한의 경우 즉(i) 平板의 두께가 無限한 경우와(ii) 타원이 원으로 reduce 되는 경우에 既存의 解와 일치됨을 보였다. 積分方程式의 解 및 應力解析은 第二報에서 다루기로 한다.

1. Introduction

Problem of determining the stress distribution in an elastic medium with flaws like elliptical or ellipsoidal form has been investigated in several papers[3, 4, 7, 8]. All these concerned with the analysis in an infinite medium and the problem when the medium has finite thickness has received little attention.

This paper concerns the determination of stress distribution in an isotropic infinite plate containing an elliptical crack when the surface of the plate is subject to a prescribed force. The stress distribution near a flat elliptical crack in an infinite medium under a uniform tension at infinity was investigated by Green and Sneddon [3]. More recently Kassir and Sih [4] have considered the problem of elliptical crack in an infinite elastic medium under uniform shear. Both of them used ellipsoidal coordinates and the Jacobian elliptic functions to solve the problem.

However, such an approach does not appear to be applicable to the problem for the medium with finite thickness. For the present analysis, double Fourier integral transform is used, and it is shown that the problem is equivalent to the solution of Fredholm integral equation of the first kind. As the ellipse reduces to a circle, it becomes an axisymmetric problem of penny-shaped crack in an infinite plate which was originally considered by Lowengrub [5]. We have shown that the present integral equation reduces to the Fredholm integral equation of the second kind which is in

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agreement with the equation derived in an alternative method.

The analysis throughout the paper is formal, and no attempt has been made to justify the interchange of various limiting processes.

2. General Equation and Solution

The crack is taken to lie in the central plane of the plate with its surfaces parallel to those of the plate. It is assumed that the deformation is due to the uniform force applied to the surface of the plate.

If we take the center of ellipse as the origin of Cartesian coordinates, and z-axis perpendicular to the surface of the crack, and x-and y-axis along the major and minor semiaxis of ellipse, respectively, the crack, then, occupies the region Ω which is governed by the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1$. The thickness of the plate will be taken to be $2h$.

If the free surface of the plate $z=h$ is pulled by a uniform tension P, the boundary conditions for the present problem can be mathematically stated as follows.

On $z=0$: $\sigma_z=0$ (x,y) inside Ω

$u_z=0$ (x,y) outside Ω

$\tau_{xz}=\tau_{yz}=0$

On $z=|h|$: $\sigma_z=P$

$\tau_{xz}=\tau_{yz}=0$

If we choose the components of the displacement vector to be $(u_x, u_y, u_z\text{-}Pz/\beta^2)$, the boundary conditions take the following form:

On $z=0$: $\sigma_z=-P$ (x,y) inside Ω (2.1)

$u_z=0$ (x,y) outside Ω (2.2)

$\tau_{xz}=\tau_{yz}=0$ (2.3)

On $z=|h|$: $\sigma_z=0$ (2.4)

$\tau_{xz}=\tau_{yz}=0$ (2.5)

where $\beta^2=\frac{\lambda'+2\mu'}{\mu'}=\frac{2(1-\nu')}{1-2\nu'}$, ν' being Poisson's ratio. If we take the unit of stress to be the rigidity modulus μ', and introduce three harmonic functions χ, φ and ψ, so that

$\nabla^2\chi=\nabla^2\varphi=\nabla^2\psi=0$

and express the component of the displacement vector in terms of them through the equations [10]

$$u_x=\frac{\partial\chi}{\partial x}+\frac{\partial\varphi}{\partial x}+(\beta^2-1)\ z\frac{\partial^2\varphi}{\partial x\partial z}+z\frac{\partial\psi}{\partial x},$$

$$u_y=\frac{\partial\chi}{\partial y}+\frac{\partial\varphi}{\partial y}+(\beta^2-1)\,z\frac{\partial^2\varphi}{\partial y\partial z}+z\frac{\partial\psi}{\partial y},\quad (2.6)$$

$$u_z=\frac{\partial\chi}{\partial z}-\beta^2\frac{\partial\varphi}{\partial z}+(\beta^2-1)\ z\frac{\partial^2\varphi}{\partial z^2}+z\frac{\partial\psi}{\partial z}-\psi,$$

It is easily verified that the stress field is given by the equations,

$$\sigma_x=-2(\beta^2-2)\frac{\partial^2\varphi}{\partial z^2}+2\frac{\partial^2\chi}{\partial x^2}+2\frac{\partial^2\varphi}{\partial x^2}$$

$$+2(\beta^2-1)\ z\frac{\partial^3\phi}{\partial x^2\partial z}+2z\frac{\partial^2\psi}{\partial x^2}-2\frac{\partial\psi}{\partial z},\quad (2.7)$$

$$\sigma_y=-2(\beta^2-2)\frac{\partial^2\varphi}{\partial z^2}+2\frac{\partial^2\chi}{\partial y^2}+2\frac{\partial^2\varphi}{\partial y^2}$$

$$+2(\beta^2-1)\,z\frac{z^3\varphi}{\partial y^2\partial z}+2z\frac{\partial^2\psi}{\partial y^2}-2\frac{\partial\psi}{\partial z},$$

$$\sigma_z=-2(\beta^2-1)\frac{\partial^2\varphi}{\partial z^2}+2\frac{\partial^2\chi}{\partial z^2}+2(\beta^2-1)\,z\frac{\partial^3\varphi}{\partial z^3}$$

$$+2z\frac{\partial^2\psi}{\partial z^2}-2\frac{\partial\psi}{\partial z};$$

$$\tau_{yz}=2(\beta^2-1)\,z\frac{\partial^3\varphi}{\partial y\partial z^2}+2z\frac{\partial^2\psi}{\partial y\partial z}+2\frac{\partial^2\chi}{\partial y\partial z}$$

$$\tau_{xz}=2(\beta^2-1)\ z\frac{\partial^3\varphi}{\partial x\partial z^2}+2z\frac{\partial^2\psi}{\partial x\partial z}+2\frac{\partial^2\chi}{\partial x\partial z}\quad (2.8)$$

$$\tau_{xy}=2\frac{\partial^2\varphi}{\partial x\partial y}+2(\beta^2-1)\,z\frac{\partial^3\varphi}{\partial x\partial y\partial z}+2z\frac{\partial^2\psi}{\partial x\partial y}$$

$$+2\frac{\partial^2\chi}{\partial x\partial y}$$

Suitable functions for φ, χ and ψ are chosen to be

$$\varphi(x,y,z)=\int_\Omega\frac{A(u,v)\,dudv}{((x-u)^2+(y-v)^2+z^2)^{1/2}}\quad (2.9)$$

$$\chi(x,y,z)=\frac{1}{2\pi}\iint_{-\infty}^{\infty}B(\xi,\eta)\ \cosh\zeta z\ e^{-i(\xi x+\eta y)}$$

$$d\xi d\eta\quad (2.10)$$

$$\psi(x,y,z)=\frac{1}{2\pi}\iint_{-\infty}^{\infty}C(\xi,\eta)\ \sinh\zeta z\ e^{-i(\xi x+\eta y)}$$

$$d\xi d\eta\quad (2.11)$$

where $\zeta=\sqrt{\xi^2+\eta^2}$

Since the limiting value of the derivative of φ is given

$$\left(\frac{\partial\varphi}{\partial z}\right)_{z\to 0+}=\begin{cases}-2\pi A(x,y) & (x,y) \text{ inside } \Omega\\ 0 & (x,y) \text{ outside} \Omega\end{cases} \tag{2.12}$$

we can immediately see that the boundary conditions (2.2) and (2.3) are automatically satisfied by the choice of the functions φ, χ, and ψ given by Eqns. (2.9) – (2.11).

It is easily shown that the boundary condition $\tau_{xz}=0$ on the surface $z=h$ yields the equation

$$3(\beta^2-1)h\int_\Omega A(u,v)\,(x-u)\left(\frac{1}{r_h^5}-\frac{5h^2}{r_h^7}\right)dudv$$
$$-\frac{i}{2\pi}\iint_{-\infty}^{\infty}\zeta\xi\{B(\xi,\eta)\sinh\zeta h+hC(\xi,\eta)$$
$$\cosh\zeta h\}\;e^{-i(\xi x+\eta y)}d\xi d\eta=0$$

where

$$r_h=((x-u)^2+(y-v)^2+h^2)^{1/2}$$

Inverting the Fourier Transform, we obtain the equation

$$B(\xi,\eta)\sinh h\zeta h+h.C(\xi,\eta)\;\cosh\zeta h$$
$$=\frac{3(\beta^2-1)h}{2\pi i\xi\zeta}\int_\Omega A(u,v)du\;dv\iint_{-\infty}^{\infty}\left(\frac{1}{r_h^5}-\frac{5h^2}{r_h^7}\right)$$
$$(x-u)\,e^{i(\xi x+\eta y)})\,dx\;dy \tag{2.13}$$

If we integrate the right hand side of Eqn. (2.13) by parts, we obtain the equation

$$\frac{(\beta^2-1)h}{2\pi\zeta}\int_\Omega A(u,v)\,du\,dv\;e^{i(\xi u+\eta v)}$$
$$\iint_{-\infty}^{\infty}\left\{\frac{1}{(x^2+y^2+h^2)^{3/2}}-\frac{3h^2}{(x^2+y^2+h^2)^{5/2}}\right\}$$
$$e^{i(\xi x+\eta y)}\;dxdy \tag{2.14}$$

If we make use of the known integrals [2]

$$\int_0^\infty\frac{\cos\xi xdx}{(x^2+y^2+h^2)^{3/2}}=\frac{\xi}{\sqrt{y^2+h^2}}K_1(\xi\sqrt{y^2+h^2})$$
$$\int_0^\infty\frac{K_1(\xi\sqrt{y^2+h^2})\cos\eta y}{\sqrt{y^2+h^2}}dy=\frac{\pi}{2h\xi}e^{-h\sqrt{\xi^2+\eta^2}}$$
$$\int_0^\infty\frac{\cos\xi xdx}{(x^2+y^2+h^2)^{5/2}}=\frac{\xi^2}{3(y^2+h^2)}K_2\;(\xi\sqrt{y^2+h^2})$$
$$\int_0^\infty\frac{K_2(\xi\sqrt{y^2+h^2})}{y^2+h^2}\cos\eta ydy=\frac{2\pi}{3}\frac{e^{-h\sqrt{\xi^2+\eta^2}}}{h^3}$$
$$(1+h\sqrt{\xi^2+\eta^2})$$

where K_1 and K_2 are modified Bessel functions of the second kind, to the inner integral of Eqn. (2.14), we find, after simplifying, one relation connecting the unknown functions $A(u,x)$, $B(\xi,\eta)$, and $C(\xi,\eta)$

$$B(\xi,\eta)\;\sinh h\zeta h+h.C(\xi,\eta)\;\cosh\zeta h$$
$$=h(\beta^2-1)e^{-\zeta h}\int_\Omega A(u,v)e^{i(\xi u+\eta v)}\;dudv \tag{2.15}$$

If we use the boundary condition $\tau_{yz}=0$ on the free surface $z=h$, we would again obtain Eqn. (2.15).

Similarly, from the bouhdary condition $\sigma_z=0$ on the free surface $z=h$, we have the equation

$$B(\xi,\eta)\zeta^2\cosh\zeta h+C(\xi,\eta)\zeta(\zeta h.\sinh\zeta h-\cosh\zeta h)$$
$$=-\frac{(\beta^2-1)}{2\pi}\int_\Omega A(u,v)\,dudv\iint_{-\infty}^{\infty}\left(\frac{1}{r_h^3}+\frac{6h^2}{r_h^5}\right.$$
$$\left.+3h^2\frac{\partial}{\partial h}\frac{1}{r_h^5}\right)e^{i(\xi x+\eta y)}\;dxdy$$
$$=-\frac{(\beta^2-1)}{2\pi}\left[\frac{2\pi}{h}e^{-\zeta h}\;(3+2\zeta h)+\sqrt{8\pi}(\sqrt{\zeta}h)^3\right.$$
$$\left.\frac{\partial}{\partial h}\{h^{-3/2}K_{3/2}(\zeta h)\}\right] \tag{2.16}$$
$$\times\int_\Omega A(u,v)\,e^{i(\xi u+\eta v)}dudv$$

If we use the relation

$$\frac{\partial}{\partial h}h^{-3/2}K_{3/2}(\zeta h)=-\zeta h^{-3/2}K_{5/2}(\zeta h)$$
$$=-\sqrt{\frac{\pi}{2\zeta h}}e^{-\zeta h}\left(1+\frac{3}{\zeta h}+\frac{3}{\zeta^2h^2}\right)$$

to eqn. (2.16), we obtain another relation between $A(u,v)$, $B(\xi,\eta)$ and $C(\xi,\eta)$

$$\zeta^2\cosh\zeta h+C(\xi,\eta)\zeta(\zeta h\;\sinh\zeta h-\cosh\zeta h)$$
$$=(\beta^2-1)\zeta\;e^{-\zeta h}(\zeta h+1)\;\int_\Omega A(u,v)\;\;e^{i(\xi u+\eta v)}dudv \tag{2.17}$$

If we solve eqns. (2.15) and (2.17) simultaneously for $B(\xi,\eta)$ and $C(\xi,\eta)$, we get

$$B(\xi,\eta)=\frac{2(\beta^2-1)\zeta h^2}{2\zeta h+\sinh\;2\zeta h}\int_\Omega A(u,v)e^{i(\xi u+\eta v)}$$
$$dudv \tag{2.18}$$

and

$$C(\xi,\eta)=-\frac{2(\beta^2-1)\;(\zeta h+e^{-\zeta h}\sinh\zeta h)}{2\zeta h+\sinh 2\zeta h}$$
$$\int_\Omega A(u,v)\,e^{i(\xi u+\eta v)}dudv \tag{2.19}$$

Now, if we substitute from eqns. (2.9)-(2.11) into the third of eqn.(2.7), and use the boundary condition (2.1), we have the equation

$$(\beta^2-1)\int_\Omega \frac{A(u,v)\,du\,dv}{r^3}+\frac{1}{2\pi}\iint_{-\infty}^{\infty}\zeta\{\zeta B(\xi,\eta)-C(\xi,\eta)\}e^{-i(\xi x+\eta y)}\,d\xi d\eta=-\frac{P}{2} \qquad (2.20)$$

where

$$r=\sqrt{(x-u)^2+(y-u)^2}$$

If we substitute from eqns. (2.18) and (2.19) into eqn. (2.20), we obtain, after simplification

$$\int_\Omega \frac{A(u,v)\,du\,dv}{r^3}-\frac{1}{2\pi}\int_\Omega A(u,v)\,du\,dv \iint_{-\infty}^{\infty}\zeta M(\zeta h)e^{-i((x-u)\xi+(y-v)\eta)}\,d\xi d\eta =-\frac{P}{2(\beta^2-1)} \qquad (2.21)$$

where

$$M(u)=\frac{-2u(u+1)+e^{-2u}-1}{2u+\sin h\,2u}$$

If we make the coordinate transformation $\xi=\zeta\cos\psi$, $\eta=\zeta\sin\psi$ and make use of the known integral [11]

$$J_0(r)=\frac{1}{2\pi}\int_0^{2\pi}e^{ir\cos\phi}\,d\phi \qquad (2.22)$$

one of the inner integrals of the second term of left hand side of eqn. (2.21) can be evaluated. Thus we have

$$\int_\Omega \frac{A(u,v)\,du\,dv}{r^3}-\int_\Omega A(u,v)\,du\,dv \int_0^{\infty}M(\zeta h)\zeta^2 J_0(\zeta r)\,d\zeta=\frac{-P}{2(\beta^2-1)}$$

The above equation can be written as

$$\nabla^2{}_1\int_\Omega A(u,v)\;k(u,v;x,y)\,dudv=-\frac{P}{2(\beta^2-1)} \qquad (2.23)$$

where

$$k(u,v{:}x,y)=\frac{1}{r}+\int_0^{\infty}M(\zeta h)J_0(\zeta r)\,d\zeta$$

with

$$\nabla^2{}_1=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

If we integrate eqn. (2.23), we obtain the desired Fredholm integral equation of the first kind

$$\int_\Omega A(u,v)\;K(u,v;x,y)\,dudv=-\frac{Px^2}{4(\beta^2-1)} \qquad (2.24)$$

where

$$K(u,v;x,y)=k(u,v;x,y)-(x^2-y^2+1)\;k(u,v;0,0)+(x^2-y^2)\,k(u,v;0,1)$$

3. Penny-shaped Crack

In the special case, when the ellipse reduces to a circle, it becomes an axisymmetric problem for the penny-shaped crack, and eqn. (2.23) can be simplified.

Since eqn. (2.21) can also be written as

$$\nabla^2{}_1\left[\int_\Omega \frac{A(u,v)}{r}du\,dv-\frac{1}{2\pi}\int_\Omega A(u,v)\,du\,dv\iint_{-\infty}^{\infty}\frac{M(\zeta h)}{\zeta}e^{-i((x-u)\xi+(y-v)\eta)}\right]d\xi d\eta =-\frac{P}{2(\beta^2-1)} \qquad (3.1)$$

if we transform the Laplacian in polar coordinates and integrate eqn. (3.1) twice and let

$x=\rho\cos\varphi$, $y=\rho\sin\varphi$

$u=\sigma\cos\theta$, $v=\sigma\sin\theta$

eqn. (3.1) can be written as, for $A(u,v)$ is the function of σ only when it is axisymmetric

$$\int_0^a A(\sigma)\int_0^{2\pi}\frac{d\omega}{\sqrt{\rho^2-2\rho\sigma\cos(\varphi-\omega)+\sigma^2}}d\sigma -\frac{1}{2\pi}\int_0^a\int_0^{2\pi}A(\sigma)\int_0^{2\pi}\int_0^{\infty}M(\zeta h)e^{-i\zeta(\rho\sin(\varphi-\psi)-\sigma\sin(\theta-\psi))}\,d\zeta d\psi d\sigma =\frac{-P\rho^2}{8(\beta^2-1)} \qquad (3.2)$$

Copson[1] has shown that the inner integral of the first term of the above equation is equal to

$$4\int_0^{\min(\rho,\sigma)}\frac{dt}{\sqrt{(\rho^2-t^2)(\sigma^2-t^2)}}$$

If we again make use of eqn. (2.22) to the second term of left hand side of eqn. (3.2), it reduces to

$$\int_0^\rho \frac{a(t)\,dt}{\sqrt{\rho^2-t^2}} - \frac{\pi}{2}\int_0^a A(\sigma)\int_0^\infty M(\zeta h) J_0(\zeta\rho) J_0(\zeta\sigma)\, d\zeta d\sigma = -\frac{P\rho^2}{8(\rho^2-1)} \qquad (3.3)$$

where

$$a(t) = \int_t^a \frac{A(\sigma)\,d\sigma}{\sqrt{\sigma^2-t^2}} \qquad (3.4)$$

Inverting above Abel type integral equation, we obtain the equation

$$a(t) + \int_0^a A(\sigma)\int_0^\infty M(\zeta h)\cos\zeta t\, J_0(\zeta\sigma)\, d\zeta d\sigma = \frac{P}{4\pi(\beta^2-1)}\frac{d}{dt}\int_0^t \frac{\rho^3 d\rho}{\sqrt{t^2-\rho^2}} \qquad (3.5)$$

In the above equation we have used the known integral [2]

$$\int_0^t \frac{\rho J_0(\zeta\rho)}{\sqrt{t^2-\rho^2}} d\rho = \frac{\sin(\zeta t)}{\zeta} \qquad (3.6)$$

From eqn. (3.4), we have

$$A(\sigma) = -\frac{2}{\pi}\frac{d}{d\sigma}\int_\sigma^a \frac{t a(t)\,dt}{\sqrt{t^2-\sigma^2}} = \frac{2}{\pi}\frac{d}{d\sigma}\int_\sigma^a a'(t)\sqrt{t^2-\sigma^2}\, dt$$

where prime denotes the differentiation with respect to the argument. We put this value of $A(\sigma)$ into eqn. (3.5), and use the Leibnitz rule.

$$\int_0^a A(\sigma) J_0(\zeta\sigma)\, d\sigma = \frac{2}{\pi}\int_0^a\int_\sigma^a \frac{\sigma a'(t)\,dt}{\sqrt{t^2-\sigma^2}} J_0(\zeta\sigma)\, d\sigma$$

Changing the order of integration in the above equation, we have

$$\frac{2}{\pi}\int_0^a a'(t)\int_0^t \frac{\sigma J_0(\sigma\zeta)}{\sqrt{t^2-\sigma^2}} d\sigma dt$$

If we make use of eqn. (3.6) and integrate by parts in the above equation, it becomes

$$-\frac{2}{\pi}\int_0^a a(t)\cos\zeta t\, dt$$

So, finally, we have the Fredholm integral equation of the second kind

$$a(\rho) - \frac{2}{\pi}\int_0^a a(t) K(t,\rho)\, dt = \frac{P\rho^2}{2\pi(\beta^2-1)} \qquad (3.7)$$

where

$$K(t,\rho) = \int_0^\infty M(\zeta h)\cos\zeta t\cos\zeta\rho\, d\zeta$$

Except range of integration and the free term, eqn. (3.7) is indentical with the integral equation for an exterior crack in an infinite plate obtained by an alternative method in [9].

4. Elliptical Crack in an Infinite Medium

When the medium is infinite, the thickness of the plate is infinite, and the second term of the left hand side of eqn. (3.1) vanishes and it reduces to

$$\nabla^2{}_1\int_\Omega \frac{A(u,v)\,dudv}{r} = -\frac{P}{2(\beta^2-1)} \qquad (4.1)$$

If we choose

$$A(u,v) = B\left(1-\frac{u^2}{a^2}-\frac{v^2}{b^2}\right)^{\frac{1}{2}}$$

the integral

$$\int_\Omega \frac{\left(1-\frac{u^2}{a^2}-\frac{v^2}{b^2}\right)^{\frac{1}{2}} dudv}{[(x-u)^2+(y-v)^2]^{1/2}}$$

is integrable and is equal to [6], for $a>b$ and $e^2=1-\frac{b^2}{a^2}$

$$\pi b\, B\left[K(e) - \frac{x^2}{a^2}\frac{K(e)-E(e)}{e^2} - \frac{y^2}{b^2}\frac{E(e)-(1-e^2)K(e)}{e^2}\right]$$

where K and E are complete elliptic integrals of the first and second kind.

Thus from eqn. (4.1), B is equal to

$$B = \frac{bP}{4\pi E(e)(\beta^2-1)}$$

Therefore from eqn. (2.12), the normal displacement over the crack is

$$u_z = \frac{bP(1-\nu')}{E(e)}\left\{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right\}^{\frac{1}{2}}$$

which is in agreement with the result of Green and Sneddon.

The solution of eqn. (2.23) and the quantities of physical interest will be considered in part II.

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