 Transactions of the Korean Society of Mechanical Engineers Vol. 5, No. 2, pp.94~100, 1981.

〈Original〉

A General Fracture Criterion Solely in Terms of Stress Intensity Factors

Kang Y. Lee*

(Received December 17, 1980)

응력 세기인수로만 표기되는 파괴기준 일반식

이 강 용

초 록

McClintock 와 Walsh 의 개념을 발전시키므로써 2차와 3차계에서의 열적 기계적 파괴 기준식을 구하였다. Griffith Theory 와 파괴역학에 기초를 두고서 전개된 응력세기인수로만 표기된 2차와 3차계의 파괴기준식은 동일한 관계식 즉, $(k_2/k_{2c})^2+k_1/k_{1c}=1$로 표시된다는 것을 보였다. 3차계에서 열적 기계적 하중하에서 응력세기 인수에 관한 일반식을 Crack 의 닫힘을 고려하여 유도하였다.

1. Introduction

Robertson [1] observed that Griffith two-dimensional failure criterion [2] does not govern the strengths of rocks. McClintock and Walsh [3] reduced this discepancy by assuming the possibility of crack/cavity closure and developing a modified Griffith two-dimensional failure criterion in terms of mechanical stresses. Recently, Paul and Mirandy [4] have developed a three-dimensional Griffith failure criterion in terms of mechanical stresses without considering crack/cavity closure. To date, relatively few attempts have been made to develop a fracture criterion in terms of stress intensity factors. Sih [5] suggested the use of the strain energy concept to predict the onset of fracture. However, his theory remains a subject of considerable controversy among several researchers. Recently, Wu [6], Shah [7] and Awaji and Sato [8] attempted to obtain empirical fracture criteria in terms of the stress intensity factors.

In this paper, by modifying a McClintock and Walsh's model, two-and three-dimensional thermo-mechanical failure criteria are developed in terms of mechanical and thermal loads as well as internal crack/cavity pressure, exclusively. A single general fracture criterion from the consideration of both two-and three-dimensional failure

*Member, Yonsei University

problems is obtained solely in terms of stress intensity factors. General expressions of stress intensity factors in three-dimensional fracture porblems are deduced under the consideration of the effect of crack closure and thermo-mechanical loading.

2. Model Formulation

The two-dimensional, thermo-mechanical-elastic model with a non-interacting, isolated crack/elliplic cavity, which is a modification of McClintock and Walsh's [3], is illustrated in Fig. 1. In this model, σ_x' and σ_y' denote the applied mechanical normal stresses, σ_{xy}' is the applied mechanical shear stress, σ_n and σ_f are the crack-closure normal and frictional shear stresses, respectively, P represents the internal crack/cavity pressure, a and b are major and minor axes of crack/cavity and T designates the temperature change associated with arbitrary thermal boundary conditions.

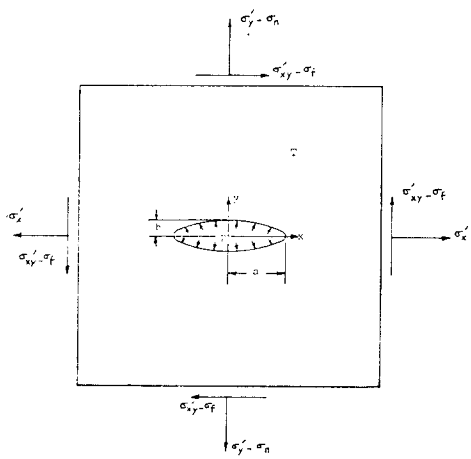

Fig. 1 Two-dimensional thermo-mechanical model.

3. General Fracture Criterion in Terms of Stress Intensity Factors

3.1. Two-Dimensional Fracture Case

By Muskhelishvili's complex variable method [9] of thermoelasticity, Lee [10] obtained the general forms of induced stress field expressions for the model in terms of curvilinear coordinates (ρ, θ) (Fig. 2) and parameters R and m which are defined by

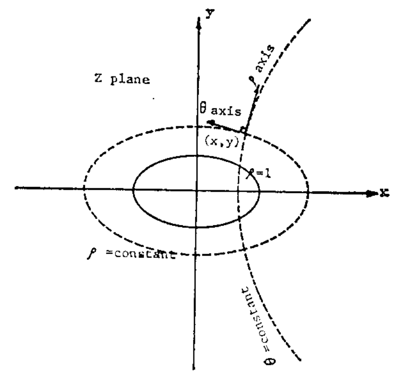

Fig. 2 Curvilinear coordinates (ρ, θ) of the point (x, y) of the z plane.

$$z=x+iy=R\left(\zeta+\frac{m}{\zeta}\right) \text{ and } \zeta=\rho e^{i\theta} \quad (1)$$

where $R>0$ and $0 \le m \le 1$

The hoop stress expression, σ_θ, is expressed on the crack/cavity surface $(\rho=1)$ in the form [10]

$$\sigma_\theta=\frac{1}{1-2m\cos 2\theta+m^2}\{(\sigma_y'-\sigma_n)(1+2\cos 2\theta-2m-m^2)+\sigma_x'(1-2\cos 2\theta+2m-m^2)+P(-3m^2+2m\cos 2\theta+1)-4(\sigma_{xy}'-\sigma_f)\sin 2\theta\}$$
$$+\frac{1}{R(1-2m\cos 2\theta+m^2)}\{3[A_R(1-m)\cos\theta+(1+m)A_I\sin\theta]-\frac{1+2m\cos 2\theta+m^2}{1-2m\cos 2\theta+m^2}[A_R(1-m)\cos\theta+A_I(1+m)\sin\theta]+\frac{2(1+m^2)}{(1-2m\cos 2\theta+m^2)^2}[A_R\cos\theta(1+m-m^2-m^3+\cos^2\theta(4m^2-4m))+A_I\sin\theta(1-m-m^2+m^3$$

$$+\sin^2\theta(4m+4m^2))]\} \qquad (2)$$

where A_R and A_I are real and imaginary part of a complex constant, A, related to thermal loading. From the definition of transformation from Cartesian to elliptical coordinates $(\bar{\rho}, \bar{\theta})$,

$$z=x+iy=a\cosh(\bar{\rho}+i\bar{\theta}) \qquad (3)$$

and eq. (1), we have, on the surface of crack/cavity $(\rho=1)$,

$$\bar{\theta}=\theta \text{ and } \frac{1-\tanh\bar{\rho}_0}{1+\tanh\bar{\rho}_0}=m \qquad (4)$$

where $\bar{\rho}_0$ corresponds to the crack surface, $\rho=1$, and characterizes the geometry of the elliptical crack.

Failure is governed by the maximum hoop stress concentration around the limiting cavity (or crack) tip (i.e. limiting values of $\bar{\rho}_0$ and $\bar{\theta}_0$) (Griffith [2], McClintock and Walsh [3] and Murrell [11]). By inserting eq. (4) into eq. (2), expressing the trigonometric functions in terms of binominal expressions and ignoring smaller order terms, the tangential stress for small values of $\bar{\rho}_0$ and $\bar{\theta}_0$ induced by the thermo-mechanical loads is given by

$$\sigma_\theta=\left(\sigma_y'-\sigma_n+P+\frac{2A_R}{a}\right)\frac{2\bar{\rho}_0}{\bar{\rho}_0^2+\bar{\theta}^2} +\left(-\sigma_{xy}'+\sigma_f+\frac{2A_I}{a}\right)\frac{2\theta}{\bar{\rho}_0^2+\bar{\theta}^2} \qquad (5)$$

Griffith theory is based on the maximum tangential stress given by (Griffith [2], McClintock and Walsh [3], Murrell [11])

$$\sigma_\theta=\frac{2S_t}{\bar{\rho}_0} \qquad (6)$$

Where S_t is a material tensile strength.

The failure crite ion for a sharp crack is derived by equating eq. (6) and the maximum tangential stress determined by eq. (5) in the form

$$L_2=-4S_t\left(1-\frac{L_1}{2S_t}\right)^{1/2} \qquad (7)$$

where

$$L_1=2\left(\sigma_y'-\sigma_n+P+\frac{2A_R}{a}\right) \qquad (8)$$

$$L_2=-2\left(\sigma_{xy}'-\sigma_f-\frac{2A_I}{a}\right) \qquad (9)$$

McClintock and Walsh's crack/cavity closing concept [3] is modified for the thermo-mechanical loading model, Fig. 1, in the form [10]

$$\sigma_n=0 \text{ for either } \sigma_y'\geq 3\alpha MT_0 \text{ or } 0\geq\sigma_y'-3\alpha MT_0\geq\sigma_c \qquad (10)$$

$$\sigma_n=\sigma_y'-3\alpha MT_0-\sigma_c \text{ for } \sigma_y'-3\alpha MT_0<\sigma_c<0 \qquad (11)$$

$$\sigma_f=-\mu_f\sigma_n \text{ for } \sigma_{xy}'\geq\sigma_f \qquad (12)$$

Where α is the coefficient of linear thermal expansion, M a bulk modulus, T_0 temperature change on the surface of the crack/cavity, σ_c the critical stress to close the crack/cavity at least at a location, and μ_f the friction coefficient of material.

Bt introducing eqs. (11) and (12) into Eqs. (7), (8) and (9), transforming the result in terms of principal stresses (σ_1, σ_3) and differentiating it with respect to the angle, Φ, between the major axis of the crack/cavity and the minimum principal stress (σ_3) axis, the orientation of the most severely stressed crack/cavity is obtained in the form

$$\tan 2\Phi=1/\mu_f \qquad (13)$$

In addition to eq. (13), we also obtain the two-dimensional thermo-mechanical failure criterion in the form

$$(\sigma_1-\sigma_3)(1+\mu_f^2)^{1/2}-\frac{4A_I}{a}+\mu_f\{\sigma_1+\sigma_3+2P-2(3\alpha MT_0+\sigma_c)\} =4S_t\left\{1-\left(3\alpha MT_0+\sigma_c+P+\frac{2A_R}{a}\right)/S_t\right\}^{1/2} \qquad (14)$$

For the case with no thermal load and internal pressure, eq. (14) is reduced to the McClintock and Walsh's modified Griffith failure criterion [3].

The stress intensity factors for model 1 and 2 (k_1 and k_2, respectively) corresponding to the model, Fig. 1, can be shown in the form [10]

$$k_1=\left(\sigma_y'-\sigma_n+P+\frac{2A_R}{a}\right)\sqrt{a} \quad (15)$$

$$k_2=\left(\sigma_{xy}'-\sigma_f-\frac{2A_I}{a}\right)\sqrt{a} \quad (16)$$

Introducing eqs. (15) and (16) into eqs. (8) and (9), eq. (7) becomes, from the concept of critical stress intensity factor (i.e. $k_1=k_{1c}$ when $k_2=0$, $k_2=k_{2c}$ when $k_1=0$),

$$\left(\frac{k_2}{k_{2c}}\right)^2+\frac{k_1}{k_{1c}}=1 \quad (17)$$

$$\text{and } k_{2c}=2k_{1c} \quad (18)$$

where k_{1c} and k_{2c} are the critical stress intensity factors for mode 1 and mode 2, respectively. It can be shown that Griffith biaxial failure criterion [2] is transformed to the same relation as eq. (17).

3.2. Three-Dimensional Fracture Case

By examining the flat ellipsoidal problem in the general three-dimensional mechanical loading at infinity as shown in Fig. 3, Paul and Mirandy [4] obtained the three-dimensional Griffith failure criterion in the form

$$(S/S_t)^2+M_1^2(N/S_t-1)=0$$
$$\text{at } \phi_1=0,\ \phi_2=\frac{1}{2}\text{ arc tan}$$
$$\{-2(1-N/S_t)^{1/2}/(N/S_t)\}$$
$$\text{and } \lambda=0 \quad (19)$$

where $M_1=2(1-\nu)+\Sigma$,

$$\Sigma=\frac{2\nu\delta(F-I)}{I(1-\delta^2)},\quad \delta=\frac{a}{b}\leq 1 \text{ (Fig. 4)}$$

F and I are complete elliptic integrals of the first and second kind, respectively, ν is the Poisson's ratio, N and S are normal and shear stresses, respectively (Fig. 3) and the angles, λ, ϕ_1 and ϕ_2 are defined in Fig. 3 and 4, respectively.

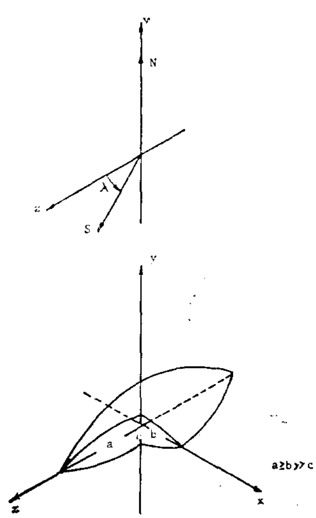

Fig. 3 Loading on flat ellipsoidal cavity.

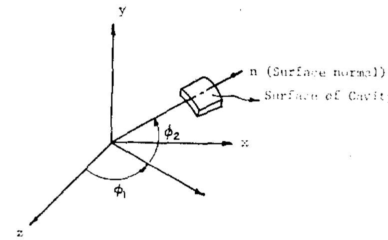

Fig. 4 Location of ellipsoid surface normal.

Mirandy and Paul [12] further deduced the stress intensity factors for each mode, k_1, k_2 and k_3, for plane elliptical crack ($c\rightarrow 0$). It can be shown that from their solutions, k_1 is maximum at $\phi_1=\pi/2$ for any ratio b/a, k_2 is maximum for $\lambda=\pi/2$ and $\phi_1=\pi/2$ for all b/a and k_3 is maximum for $\lambda=\pi/2$ and $\phi_1=0$ for all b/a.

These maximum stress intensity factors with respect to λ and ϕ_1 are given by

$$k_1=\frac{N\sqrt{b}}{I} \quad (20)$$

$$k_2=\frac{2S\sqrt{b}}{IM_2} \quad (21)$$

$$k_3=\frac{2(1-\nu)Sb}{IM_2\sqrt{a}} \quad (22)$$

where $M_2=2-\Sigma$

By introducing eqs. (20) and (21) into eq. (19) and applying the definition of critical stress intensity factors, we obtain the same equation as eq. (17) and

$$k_{2c}/k_{1c}=2M_1/M_2 \tag{23}$$

Similarly, when eq. (22) is utilized rather than eq. (21) in the above procedure and the relation between k_2 and k_3 from eqs. (21) and (22) is introduced into the result, the same equation as eq. (17) is deduced. In addition to that, the following relationship can be shown

$$k_{3c}/k_{1c}=2(1-\nu)M_1\sqrt{b}/(M_2\sqrt{a}) \tag{24}$$

We emphasize again to have shown above that the fracture criteria for two-and three-dimensional problems are ended up in the single relation, eq. (17). The expressions for stress intensity factors (k_1, k_2 and k_3) associated with a very flat ellipsoidal cavity under seperate three-dimensional mechancal and thermal loading, which are developed by Kassir and Sih [13, 14, 15] and Mirandy and Paul [12], can be made in terms of the parametric angle of the ellipse, η, which is related to the coordinates, x, z and ϕ_1 (Fig. 3 and 4) by the relations

$$\begin{aligned} &x=b\cos\eta \text{ and } z=a\sin\eta \\ &\sin\phi_1=a\sin\eta/\{a^2\sin^2\eta+b^2\cos^2\eta\}^{1/2} \\ &\cos\phi_1=b\cos\eta/\{a^2\sin^2\eta+b^2\cos^2\eta\}^{1/2} \end{aligned} \tag{25}$$

By comparing their solutions for the corresponding stress intensity factors for each mode, we obtain the following stress intensity equivalence between mechanical and thermal loadings

N corresponds to A_t (26)

S corresponds to $2\mu B_t I M_1 a\cos\phi_1/\{b^2\cos\lambda(b^2\sin^2\phi_1+a^2\cos^2\phi_1)^{1/2}\}$, which also corresponds to $2\mu C_t I M_2\sin\phi_1/\{a\sin\lambda(b^2\sin^2\phi_1+a^2\cos^2\phi_1)\}^{1/2}$ (27)

where A_t is a thermal loading constant with the dimensions of stress, which is related to three-dimensional crack surface temperature change. The B_t and C_t have the dimensions of area and are related to the three-dimensional crack surface temperature gradient. μ is a shear modulus. The stress intensity factors for the various loadings lend themselves to superposition as shown in eqs. (15) and (16). Under arbitrary thermo-mechanical loading, the general forms of the stress intensity factors (k_1, k_2 and k_3) with the effect of crack-closure can be obtained from eqs. (26) and (27) with Mirandy and Paul's solutions [12] under general mechanical loading in the form

$$k_1=\left(\frac{b}{a}\right)^{1/2}\left(\frac{1}{I}\right)\frac{(ab)^{1/2}(N-\sigma_n+P+A_t)}{\{a^2\cos^2\phi_1+b^2\sin^2\phi_1\}^{1/4}} \tag{28}$$

$$k_2=\frac{2k_1}{(N-\sigma_n+P+A_t)}\left\{\frac{\cos\phi_1\cos\lambda}{M_1}+\frac{\sin\phi_1\sin\lambda}{M_2}\right\}\cdot\left\{S-\sigma_f+\frac{2\mu C_t I M_2\sin\phi_1}{a\sin\lambda(b^2\sin^2\phi_1+a^2\cos^2\phi_1)^{1/2}}\right\} \tag{29}$$

$$k_3=\frac{2k_1(1-\nu)}{N-\sigma_n+P+A_t}\left\{\frac{-\sin\phi_1\cos\lambda}{M_1}+\frac{\cos\phi_1\sin\lambda}{M_2}\right\}\cdot\left\{S-\sigma_f+\frac{2\mu C_t I M_2\sin\phi_1}{a\sin\lambda(b^2\sin^2\phi_1+a^2\cos^2\phi_1)^{1/2}}\right\} \tag{30}$$

Introducing the stresses, σ_n and σ_f defined by eqs. (11) and (12) into eqs. (28), (29) and (30), we get

$$k_1=\left(\frac{b}{a}\right)^{1/2}\left(\frac{1}{I}\right)\frac{(ab)^{1/2}(3\alpha MT_0+\sigma_c+P+A_t)}{(a^2\cos^2\phi_1+b^2\sin^2\phi_1)^{1/4}} \tag{31}$$

$$k_2=\frac{2k_1}{(3\alpha MT_0+\sigma_c+P+A_t)}\left\{\frac{\cos\phi_1\cos\lambda}{M_1}+\frac{\sin\phi_1\sin\lambda}{M_2}\right\}\cdot\left\{S+\mu_f(N-3\alpha MT_0-\sigma_c)+\frac{2\mu C_t I M_2\sin\phi_1}{a\sin\lambda(b^2\sin^2\phi_1+a^2\cos^2\phi_1)^{1/2}}\right\} \tag{32}$$

$$k_3=\frac{2k_1(1-\nu)}{(3\alpha MT_0+\sigma_c+P+A_t)}\left\{\frac{-\sin\phi_1\cos\lambda}{M_1}+\frac{\cos\phi_1\sin\lambda}{M_2}\right\}\left\{S+\mu_f(N-3\alpha MT_0-\sigma_c)+\frac{2\mu C_t IM_2\sin\phi_1}{a\sin\lambda(b^2\sin^2\phi_1+a^2\cos^2\phi_1)^{1/2}}\right\} \qquad (33)$$

These equations express the stress intensity factors for a partially closed elliptic crack. By introducing the maximum values of k_1 and k_2 from eqs. (31) and (32) with respect to λ and ϕ_1 as well as $k_{1c}=S_t\sqrt{b}/l$ and eq. (23) into eq. (17) and following the same procedure as in the derivation of the two-dimenstional thermo-mechanical failure criterion, eq. (14), we obtain the same expression as eq. (13) for the orientation of the most severely stressed crack/cavity and the following three-dimensional thermo-mechanical failure criterion

$$(\sigma_1-\sigma_3)(1+\mu_f^2)^{1/2}-2D_t+\mu_f\{\sigma_1+\sigma_3+2P-2(3\alpha MT_0+\sigma_c)\}=2S_tM_1\left\{1-\frac{3\alpha MT_0+\sigma_c+P+A_t}{S_t}\right\}^{1/2} \qquad (34)$$

where $D_t=-2\mu C_t IM_2/ab$

3.3. Discussions

The general fracture criterion of the two- and three-dimensional crack problems, eq. (17), is plotted in terms of k_2/k_{2c} and k_1/k_{1c} in Fig. 5 and compared with Awaji and Sato's experimental data [8] for the fracture measurement of graphite and marble and also Shah's empirical criterion [7] for fracturing 4340 steel. Although Awaji and Sato have indicated that Shah's data are insufficient for characterizing the empirical equation, it is felt that Awaji and Sato's data also suffers from the similar lack of data.

Fig. 5 also shows that eq. (17) is identical to Wu's empirical relation [6] for both

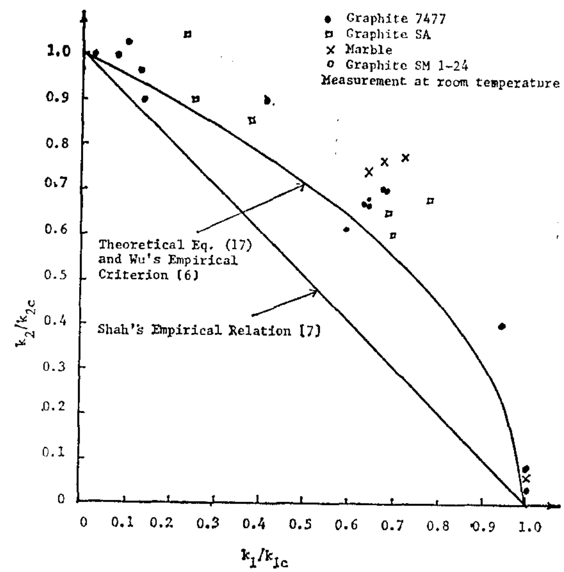

Fig. 5 Comparisions between theoretical general fracture criterion, eq. (17), experimental data and empirical relations.

balsa wood and unidirectional fiber-glass-reinforced plastic plates with a crack along the principal direction of elastic symmetry. From comparision, it is concluded that eq. (17) provides a rational and physical basis since it is derived from Griffith's failure criterion.

References

1. Robertson, E.C. (1955), "Experimental Study of the Strength of Rocks," Bulletin of Geological Society of America, Vol. 66, pp. 1275-1314
2. Griffith, A.A. (1924), "Theory of Rupture," Proceedings of the First International Congress for Applied Mechanics, pp. 55-63.
3. McClintock, F.A. and Walsh, J.B. (1962), "Friction on Griffith Cracks in Rocks under Pressure," Fourth U.S. National Congress for Applied Mechanics, Berkeley, California, pp. 1015-1021.
4. Paul, B. and Mirandy, L. (1976), "An

Improved Fracture Criterion for Three-Dimensional Stress States," Transactions of ASME, Journal of Engineering Materials and Technology, Vol. 98, pp. 159-163.

5. Sih, G.C. (1974), "Strain-Energy-Density Factor Applied to Mixed Mode Crack Problems," International Journal of Fracture Mechanics, Vol. 10. No. 3. pp. 305-321.
6. Wu, E.M. (1967), "Application of Fracture Mechanics to Anisotropic Plates," Transactions of ASME, Journal of Applied Mechanics, December, pp. 967-974.
7. Shah, R.C. (1974), "Fracture Under Combined Modes in 4340 Steel," ASTM Special Technical Publicaton No. 560, pp. 29-52.
8. Awaji, H. and Sato, S. (1978), "Combined Mode Fracture Toughness Measurement by the Disk Test," Transactions of ASME, Journal of Engineering Materials and Technology, Vol. 100, April, pp. 175-182.
9. Muskhelishvili, N.I. (1953), Some Basic Problems of Mathematical Theory of Elasticity, English Translation, P. Noordhoff and Company, New York.
10. Lee, K.Y. and Advani, S.H. (1978), "Stress and Displacement Responses of Porous-Permeable Media under Thermo-Mechanical Loading," Proceedings of the 19th U.S. Symposium on Rock Mechanics, Reno, Nevada, May 1-3, pp. 268-273.
11. Murrell, S.A.F. (1964), "The Theory of the Propagation of Elliptical Griffith Cracks under Various Conditions of Plane Strain or Plane Stress: Part I," British Journal of Applied Physics, Vol. 15, pp. 1195-1210.
12. Mirandy, L. and Paul, B. (1976), "Stress at the Surface of a Flat Three-Dimensional Elliptical Cavity," Transactions of ASME, Journal of Engineering Materials and Technology, Vol. 98, pp. 164-172
13. Kassir, M. and Sih, G. (1966), "Three Dimensional Stress Distribution around an Elliptic Crack under Arbitrary Loadings," Transactions of ASME, Journal of Applied Mechanics, Vol. 33, pp. 601-611.
14. Kassir, M.K. and Sih, G.C. (1967), "Three Dimensional Thermo-Elastic Problems of Planes of Discontinuities or Cracks in Solids," Developments in Theoretical and Applied Mechanics, Edited by W.A. Shaw, Pergamon Press, pp. 117-146.
15. Kassir, M.K. and Sih, G.C. (1975), Mechanics of Fracture 2—Three Dimensional Crack Problems, Noordhoff International Publishing, Leyden.