ON THE STRUCTURE OF THE HALL–YAMADA SEMIGROUPS

By Dong Kie Kim

1. Introduction

An orthodox semigroup is defined as a regular semigroup in which the idempotents form a subsemigroup. The class of orthodox semigroups thus includes both the class of inverse semigroups and the class of bands. Fantham [6], Yamada [15] and Petrich [12] have studied the case where the semigroup is also a union of groups. Specializing in another direction, Yamada [16] have studied the case where the band of idempotents of the semigroup is normal. Recently, the structure of orthodox semigroups in general has been clarified by Yamada [17] and Hall [8] independently. More recently, Hall [9] has generalized the Munn semigroup further in the case of a general regular semigroup.

Let $S$ be an orthodox semigroup with band $B$ of idempotents. The Hall semigroup $\mathcal{H}(B)$ plays an important role in the structure theory to be discussed in this paper. Many of the idea involved are from Yamada’s paper [19].

Our main theorem in this paper is Theorem 3.3. This theorem asserts that the Hall–Yamada semigroup $S=\mathcal{H}(B, T, \phi)$ is an orthodox semigroup whose band of idempotents is isomorphic to $B$ and that if $\gamma$ is the minimum inverse semigroup congruence on $S$ then $S/\gamma \cong T$. Conversely, if $S$ is an orthodox semigroup whose band of idempotents is $B$ then there is an idempotent-separating homomorphism $\theta : S/\gamma \to W_B/\gamma_1$ whose range contains all the idempotents of $W_B/\gamma_1$ and such that $S=\mathcal{H}(B, S/\gamma, \theta)$, where $\gamma_1$ is the minimum inverse semigroup congruence on the Hall semigroup $W_B$ of $B$.

In section 2 we discuss basic properties of semigroups which are essential to understand our main theorem.

The notation and the terminology in this paper are standard. They are taken from [4]. Let $\rho$ be a congruence on a semigroup $S$. Then $S/\rho$ denotes the factor semigroup of $S$ modulo $\rho$, and $\rho^A : S \to S/\rho$ denotes the natural homomorphism of $S$ onto $S/\rho$. Let $X$ be a set. Then $\mathcal{G}(X)$ means the semigroup of all transformations of $X$, and $\mathcal{G}(X)$ means the semigroup
of all partial transformations of $X$. The group of all permutations of $X$ is denoted by $\mathcal{G}(X)$. By the (left–right) dual $S^*$ of a semigroup $S$ we mean the semigroup $(S, \circ)$, the elements of which are the same as those of $S$, and in which the binary operation $\circ$ is defined by $a \circ b = ba$ for all $a, b$ in $S$.

2. Preliminaries

In this section we shall state several propositions which are useful in the next section. The proofs of propositions shall be omitted.

An element $a$ of a semigroup $S$ is called regular if $a \in aSa$. A semigroup $S$ is called regular if every element of $S$ is regular. Two elements $a$ and $b$ of a semigroup $S$ are said to be inverses of each other if $aba = a$ and $bab = b$. By an inverse semigroup we mean a semigroup in which every element has a unique inverse. A band is a semigroup in which every element is idempotent.

**Definition 2.1.** For each element $a$ of a semigroup $S$, let

$$V(a) = \{b \in S : b \text{ is an inverse of } a\}$$

It is easy to prove the following proposition.

**Proposition 2.2** Let $S$ be an orthodox semigroup. Then the relation $\gamma$ on $S$ defined by

$$\gamma = \{(x, y) \in S \times S : V(x) = V(y)\}$$

is a congruence on $S$.

Moreover, it is the smallest inverse semigroup congruence on $S$.

**Definition 2.3.** Let $S$ be a semigroup. Define relations $L$ and $R$ on $S$ by

$$L = \{(a, b) \in S \times S : a \cup Sa = b \cup Sb\},$$

$$R = \{(a, b) \in S \times S : a \cup aS = b \cup bS\}.$$  

Then $L$ and $R$ are a right and left congruence, respectively.

The relations $L$ and $R$ commute and so the relation $\mathcal{D} = L \circ R = R \circ L$ is the smallest equivalence relation containing both $L$ and $R$. Moreover, the relation $\mathcal{H} = L \cap R$ is an equivalence relation.

We denote the $L$-class, the $R$-class, the $D$-class and the $H$-class containing an element $a$ by $L_a$, $R_a$, $D_a$ and $H_a$, respectively.

It is known that a congruence $\rho$ on a regular semigroup $S$ is idempotent–separating if and only if $\rho \subseteq \mathcal{H}$. In particular, the congruence $\mathcal{K}^\#$, the largest congruence contained in $\mathcal{K}$, is the maximum idempotent–separating congruence on a regular semigroup $S$. Moreover, the following proposition holds.
PROPOSITION 2.4. Let $S$ be an inverse semigroup with semilattice $E$ of idempotents. Then the relation
$$\mu = \{(a, b) \in S \times S : a^{-1}ea = b^{-1}eb \text{ for all } e \in E\}$$
is the maximum idempotent-separating congruence on $S$.

Let $S$ be an orthodox semigroup with semilattice $B$ of idempotents. And let $\gamma$ be the congruence on $S$ defined in Proposition 2.2 and let
$$\varepsilon = \gamma \cup (B \times B).$$
Then $\varepsilon$ is a congruence on $B$, and there exists a monomorphism
$$\eta : B/\varepsilon \to S/\gamma$$
which commutes the following diagram.

Therefore, the semilattice of idempotents of the maximum inverse semigroup homomorphic image of an orthodox semigroup $S$ is isomorphic to the maximum semilattice homomorphic image of the band $B$ of idempotents of $S$. Furthermore, the following holds.

PROPOSITION 2.5. Let $S$ be an orthodox semigroup with band $B$ of idempotents. If $\mu = \mathcal{H}$ is the maximum idempotent-separating congruence on $S$, then $(a, b) \in \mu$ if and only if there exist $a' \in \mathcal{V}(a)$ and $b' \in \mathcal{V}(b)$ such that
$$a'xa = b'xb \text{ and } axa' = bxb'.$$

By a representation of a semigroup $S$ by partial transformations of a set $X$ we mean a homomorphism $\varphi : S \to \mathcal{P}(X)$ of $S$ into $\mathcal{P}(X)$, where $\mathcal{P}(X)$ is the semigroup of all partial transformations of $X$. It is known that a mapping $\varphi : S \to \mathcal{P}(S)$ which associates with each $a$ of $S$ an element $\delta_a$ defined by
$$\delta_a = \{(x, y) \in S \times S : y = xa \text{ and } (x, y) \in \mathcal{R}\}$$
is a representation of a semigroup $S$ by partial transformations. But this representation is not in general faithful. In the case of a regular semigroup we can overcome this disadvantage by simultaneously considering the (left-right) dual of $\delta_a$. Let $\mathcal{P}(S)^* S$ denote the dual semigroup of $\mathcal{P}(S)$. We have the following result.

PROPOSITION 2.6. Let $S$ be a regular semigroup. For each $a$ in $S$ define
$$\delta_a = \{(x, y) \in S \times S : y = xa \text{ and } (x, y) \in \mathcal{R}\},$$
$$\gamma_a = \{(x, y) \in S \times S : y = ax \text{ and } (x, y) \in \mathcal{L}\}.$$Then the representation $\alpha : S \to \mathcal{P}(S) \times \mathcal{P}(S)^*$ defined by $a\alpha = (\delta_a, \gamma_a)$ is faithful.
Let $S$ be an orthodox semigroup with band $B$ of idempotents. Then we can define, for each $a$ in $S$, a mapping $\rho_a : B/\mathcal{E} \to B/\mathcal{E}$ by

$$L_a\rho_a = L_{a'}xa,$$

where $a'$ is an arbitrarily chosen inverse of $a$. In particular, if $e \in B$ then we have $L_a\rho_e = L_{xe}$.

Note that if $S$ is an inverse semigroup (so that $B$ is a semilattice) then $L_a = \{x\}$ and $L_{a'}xa = \{a^{-1}xa\}$. By dual arguments we can define, for each $a$ in $S$, a mapping $\lambda_a : B/\mathcal{R} \to B/\mathcal{R}$ by

$$R_a\lambda_a = R_{axa'},$$

where $a'$ is an arbitrarily chosen inverse of $a$.

Now we have the following result.

**Proposition 2.7.** Let $S$ be an orthodox semigroup with band $B$ of idempotents. Let $\xi : S \to \mathcal{E}(B/\mathcal{E}) \times \mathcal{E}^*(B/\mathcal{R})$ be a mapping defined by

$$a\xi = (\rho_a, \lambda_a),$$

where $\rho_a$ and $\lambda_a$ are given by

$$L_a\rho_a = L_{a'}xa, \quad R_a\lambda_a = R_{axa'} \quad (x \in B).$$

Then $\xi$ is a homomorphism whose kernel is the maximum idempotent-separating congruence $\mu$ on $S$.

Let $B$ be a band and $E$ a semilattice of $B$. For each $e$ in $B$ define $Ee = \{x \in E : x \leq e\}$. Then it is easy to see that $eBe = Be$. We denote $eBe$ by $\langle e \rangle$. Note that for all $x, y$ in $B$

$$\langle x \rangle = \langle y \rangle \iff x = y.$$

We define

$$\mathcal{U} = \{(e, f) \in B \times B : \langle e \rangle \cong \langle f \rangle\}$$

and write $W_{e,f}$ for the set of all isomorphisms of $\langle e \rangle$ onto $\langle f \rangle$. Note that if $g \in \langle e \rangle$ and $\alpha \in W_{e,f}$ then

$$\langle g \rangle \alpha = \langle ga \rangle \quad \text{and} \quad ea = f.$$

If $(e, f) \in \mathcal{U}$ and $\alpha \in W_{e,f}$, we may define $\alpha_l \in \mathcal{E}(B/\mathcal{E})$ and $\alpha_r \in \mathcal{E}^*(B/\mathcal{R})$ by

$$L_a\alpha_l = L_{xa}, \quad R_a\alpha_r = R_{xa} \quad (x \in \langle e \rangle).$$

Now let $S$ be an orthodox semigroup with band $B$ of idempotents. Let $a \in S$ and $a' \in V(a)$. Denoting $aa'$ by $e$ and $a'a$ by $f$, we observe that the mapping $\rho_a \in \mathcal{E}(B/\mathcal{E})$ defined in Proposition 2.7 may be expressed as $\rho_e \theta_l$, where $\theta$ is an element of $W_{e,f}$ which is the mapping given by

$$x\theta = a'xa \quad (x \in \langle e \rangle).$$

Similarly, the mapping $\lambda_a \in \mathcal{E}(B/\mathcal{R})$ defined in Proposition 2.7 may be expressed as $\lambda_f \theta_r^{-1}$. The range of the mapping $\xi : S \to \mathcal{E}(B/\mathcal{E}) \times \mathcal{E}^*(B/\mathcal{R})$ defined by $a\xi = (\rho_a, \lambda_a)$ is thus contained in the subset
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\[ W_B = \{ (e, f) \in W_{e,f} : (e, f) \in \mathcal{U} \} \]

of \( \mathcal{U}(B/\mathcal{L}) \times \mathcal{U}^*(B/\mathcal{R}) \). We say that \( W_B \) is the Hall semigroup of the band \( B \). Now we have the following result.

**Proposition 2.8.** Let \( B \) be a band and let
\[ W_B = \{ (e, f) \in W_{e,f} : (e, f) \in \mathcal{U} \} . \]
Then
1. \( W_B \) is a subsemigroup of \( \mathcal{U}(B/\mathcal{L}) \times \mathcal{U}^*(B/\mathcal{R}) \).
2. \( W_B \) is orthodox, with band of idempotents \( B^* = \{ (e, f) : e \in B \} \) isomorphic to \( B \).
3. If \( B^* \) is identified with \( B \), then, in \( W_B \),
\[ \mathcal{U} \cup (B \times B) = \mathcal{U} . \]

3. **Main Theorem**

Let \( S \) be an orthodox semigroup with band \( B \) of idempotents. Then, by Proposition 2.7, the mapping \( \xi : a \rightarrow (\rho_a, \lambda_a) \) of \( S \) onto \( W_B \) is not in general one-one. Indeed its kernel is \( \mu \). However, since we have
\[ \gamma \cap \mu \subseteq \gamma \cap \mathcal{M} = I_S, \]
the homomorphism \( \eta : S \rightarrow W_B \) defined by
\[ a\eta = (\rho_a, \lambda_a, \alpha) \] (1)
is one-one (see Proposition 2.2.),

If \( \gamma_1 \) is the minimum inverse semigroup congruence on \( W_B \), then \( \xi_{\gamma_1}^{-1} \) is a homomorphism of \( S \) into the inverse semigroup \( W_B / \gamma_1 \) which must factor through \( S / \gamma \) in accordance with the commutative diagram

\[ \begin{array}{ccc}
S / \gamma & \longrightarrow & W_B / \gamma_1 \\
\downarrow \xi_{\gamma} & & \downarrow \xi_{\gamma_1}^{-1} \\
S & \longrightarrow & W_B \\
\end{array} \] (2)

The homomorphism \( \theta \) is uniquely determined, and we have the following lemma.

**Lemma 3.1.** The homomorphism \( \theta \) is idempotent-separating. The range of \( \theta \) contains all the idempotents of \( W_B / \gamma_1 \).

**Proof.** Let \( e\gamma \) and \( f \gamma \) be idempotents in \( S / \gamma \) (where \( e, f \in B \)) and suppose that \( (e\gamma)\theta = (f \gamma) \theta \). Then \( e\xi_{\gamma_1} = f \xi_{\gamma_1} \), that is, \( (e\xi, f \xi) \in \gamma_1 \cap (B^* \times B^*) \), where \( B^* \) is the band of idempotents of \( W_B \). It follows that the idempotents \( e\xi, f \xi \) in \( W_B \) are \( \mathcal{Q} \)-equivalent in \( B^* \). Since \( \xi | B \) is an isomorphism of \( B \) onto the band \( B^* \), it follows that \( e \) and \( f \) are \( \mathcal{Q} \)-equivalent in \( B \). Hence we have \( e\gamma = f \gamma \).
Any idempotent in \( W_B/\gamma_1 \) is expressible as \((\rho_a, \lambda_a)\gamma_1\), where \((\rho_a, \lambda_a)\) is an idempotent in \( W_B \). Thus it is expressible as \( e\xi \gamma_1 \) for some idempotent \( e \) in \( S \). The commutativity of diagram (2) then enables us to express our idempotent as \((e\gamma)\theta\). Hence every idempotent in \( W_B/\gamma_1 \) lies in the range of \( \theta \).

Any element \((\rho_a, \lambda_a, a\gamma)\) in the range of \( \gamma \) has the property that
\[
(\rho_a, \lambda_a)\gamma_1 = a\xi \gamma_1 = a\gamma \theta = (a\gamma)\theta.
\]

Conversely, we shall show that if \((x, a\gamma) \in W_B \times S/\gamma \) is an element such that \( x\gamma_1 = (a\gamma)\theta \) then \((x, a\gamma) = b\eta \) for some \( b \) in \( S \). In other words, we establish

**Proposition 3.2.** Let \( S \) be an orthodox semigroup with band \( B \) of idempotents. The mapping \( \gamma : S \rightarrow W_B \times S/\gamma \) defined by (1) is an isomorphism of \( S \) onto

\[
\{(x, a\gamma) \in W_B \times S/\gamma : x\gamma_1 = (a\gamma)\theta, \}
\]

the spined product of \( W_B \) and \( S/\gamma \) with respect to \( W_B/\gamma_1, \ r_1\xi \) and \( \theta \).

**Proof.** It remains to show that \( \gamma \) is onto. Let \((x, a\gamma) \in W_B \times S/\gamma \) such that \( x\gamma_1 = (a\gamma)\theta \). Then \( x\gamma_1 = (a\gamma)\theta = (\rho_a, \lambda_a)\gamma_1 \) so that \( V(x) = V(\rho_a, \lambda_a) \) in \( W_B \). Now for any inverse \( c \) of \( a \) in \( S \) it is easy to verify that \((\rho_c, \lambda_c) \in V(\rho_a, \lambda_a) \) in \( W_B \). Hence \((\rho_c, \lambda_c) \in V(x) \) and so both \((\rho_c, \lambda_c)x \) and \( x(\rho_c, \lambda_c) \) are idempotents in \( W_B \). Therefore, there exist \( e, f \) in \( B \) such that

\[
(\rho_c, \lambda_c)x = (\rho_c, \lambda_c), \quad x(\rho_c, \lambda_c) = (\rho_f, \lambda_f).
\]

As a consequence we have that

\[
(\rho_c, \lambda_c) \mathcal{R} (\rho_c, \lambda_c), \quad (\rho_c, \lambda_c) \mathcal{L} (\rho_f, \lambda_f)
\]

in \( W_B \). That is, \( e\xi \mathcal{R} c\xi \) and \( c\xi \mathcal{L} f\xi \) in \( W_B \). Examining the first of these, we deduce that \( e\xi \) and \( c\xi \) are \( \mathcal{R} \)-equivalent in \( S\xi \). Thus there exist \( u, v \) in \( S \) such that

\[
e\xi = (c\xi)(u\xi), \quad c\xi = (e\xi)(v\xi),
\]

that is, such that

\[
(e, cu) \in \xi \circ \xi^{-1}, \quad (c, ev) \in \xi \circ \xi^{-1}^{-1}
\]

Now \( \xi \circ \xi^{-1} = \mu \subseteq \mathcal{R} \subseteq \mathcal{K} \) and so there exist \( x \) and \( y \) in \( S \) such that

\[
e = cu x, \quad c = ev y.
\]

We conclude that \( e\mathcal{R}c \) in \( S \). Similarly, \( c\mathcal{L}e \) in \( S \).

Now it assures us that the \( \mathcal{K} \)-class \( L_e \cap R_e \) contains an inverse \( b \) of \( c \). It follows that in \( W_B \) the element \((\rho_b, \lambda_b)\) is an inverse of \((\rho_c, \lambda_c)\) and that it is \( \mathcal{L} \)-equivalent to \((\rho_c, \lambda_c)\) and \( \mathcal{K} \)-equivalent to \((\rho_f, \lambda_f)\). Since \( x \) also has these properties we conclude that \( x = (\rho_b, \lambda_b) \). Note that \( b\gamma \) and \( a\gamma \) are both inverses of \( e\gamma \) in the inverse semigroup \( S/\gamma \). Hence \( b\gamma = a\gamma \) and so

\[
(x, a\gamma) = ((\rho_b, \lambda_b), b\gamma) = b\eta.
\]
It is natural to make the following construction. Let \( B \) be a band and let \( T \) be an inverse semigroup whose semilattice of idempotents is isomorphic to \( B/\varepsilon \). Let \( \gamma_1 \) be the minimum inverse semigroup congruence on the Hall semigroup \( W_B \) of \( B \). Then \( W_B/\gamma_1 \) is an inverse semigroup whose semilattice of idempotents is isomorphic to \( B/\varepsilon \). Let \( \Psi : T \rightarrow W_B/\gamma_1 \) be an idempotent-separating homomorphism whose range contains all the idempotents of \( W_B/\gamma_1 \). Then we denote the spined product

\[
S = \{(x, t) \in W_B \times T : x\gamma_1 = t\Psi \}
\]

(3)
of \( W_B \) and \( T \) with respect to \( W_B/\gamma_1, \gamma_1^3 \) and \( \Psi \) by \( \mathcal{H}(B, T, \Psi) \). This semigroup \( \mathcal{H}(B, T, \Psi) \) is called the Hall-Yamada semigroup determined by the band \( B \), the inverse semigroup \( T \) and the idempotent-separating homomorphism \( \Psi \).

**Theorem 3.3.** Let \( B, T, \gamma_1 \) and \( \Psi \) be as above. Then the Hall-Yamada semigroup \( S = \mathcal{H}(B, T, \Psi) \) is an orthodox semigroup whose band of idempotents is isomorphic to \( B \). If \( \gamma \) is the minimum inverse semigroup congruence on \( S \), then \( S/\gamma \cong T \).

Conversely, if \( S \) is an orthodox semigroup whose band of idempotents is \( B \), then there exists an idempotent-separating homomorphism \( \theta : S/\gamma \rightarrow W_B/\gamma_1 \) whose range contains all the idempotents of \( W_B/\gamma_1 \) and such that \( S = \mathcal{H}(B, S/\gamma, \theta) \).

**Proof.** Note that the second half of this theorem is a restatement of Lemma 3.1 and Proposition 3.2.

To prove the first half we shall show that \( S = \mathcal{H}(B, T, \Psi) \) is regular. It is obvious that \( W_B \times T \) is regular; indeed we can say that the set of inverses of an element \((x, t)\) in \( W_B \times T \) is \( V(x) \times \{t^{-1}\} \). If the element \((x, t)\) is in \( S \), that is, if \( t\Psi = t_1 \), then for every \( x' \) in \( V(x) \) the elements \( x'\gamma_1 \) and \( t^{-1}\Psi \) are both inverses of the element \( x\gamma_1 = t\Psi \) of the inverse semigroup \( W_B/\gamma_1 \); hence \( x'\gamma_1 = t^{-1}\Psi \) and so \((x', t^{-1}) \in S \). Thus \( S \) is a regular subsemigroup of \( W_B \times T \), and we have shown moreover that the set of inverses of an element \((x, t)\) of \( S \) is \( V(x) \times \{t^{-1}\} \).

That \( S \) is orthodox follows immediately from the fact that \( W_B \) and \( T \) are orthodox and from the fact that an element \((x, t)\) of \( W_B \times T \) is idempotent if and only if \( x \) is an idempotent of \( W_B \) and \( t \) is an idempotent of \( T \).

Let \( \mathcal{B} \) be the band of idempotents of \( S \). We know that the idempotents of \( W_B \) form a band \( B^* \) isomorphic to \( B \); indeed the mapping \( \xi : B \rightarrow (\rho, \lambda) \) is an isomorphism of \( B \) onto \( B^\star \). Denoting the inverse of \( \xi \) by \( \kappa : B^\star \rightarrow B \), we define a mapping \( \zeta : \mathcal{B} \rightarrow B \) by

\[
(x, t)\zeta = x\kappa \quad ((x, t) \in \mathcal{B}).
\]
It is clear that $\zeta$ is a homomorphism. To see that it is onto, note that for any $e$ in $B$ the element $(\rho_e, \lambda_e)\gamma_1^h$ is an idempotent in $W_B/\gamma_1$, and so there is a unique idempotent $g$ in $T$ such that $g\mathcal{T} = (\rho_e, \lambda_e)\gamma_1^h$. Then $((\rho_e, \lambda_e)g) \in \overline{B}$ and has image $e$ under $\zeta$.

To show that $\zeta$ is one-one, suppose that the elements $(x, t), (y, u)$ in $\overline{B}$ are such that $(x, t)\zeta = (y, u)\zeta$. Then $x\kappa = y\kappa$ and so $x = y$ since $\kappa$ is an isomorphism. Hence $x\gamma_1^h = y\gamma_1^h$ and so $t\mathcal{T} = u\mathcal{T}$ by the definition formula (3) of $S$. But $t$ and $u$ are idempotents of $T$ and so, since $\mathcal{T}$ is idempotent-separating, $t = u$. Thus $(x, t) = (y, u)$, and we conclude that $\zeta$ is an isomorphism of $\overline{B}$ onto $B$.

It is easy to see that $\pi : (x, t) \mapsto t$ is a homomorphism of $S$ into the inverse semigroup $T$. In fact, $\pi$ maps onto $T$, since $\gamma_1^h$ maps $W_B$ onto $W_B/\gamma_1$ and so for every $t$ in $T$ there is an element $x$ in $W_B$ such that $x\gamma_1^h = t\mathcal{T}$, that is, such that $(x, t) \in S$. If $\gamma$ is the minimum inverse semigroup congruence on $S$, it follows that $\gamma \subseteq \pi \circ \pi^{-1}$ and that there is a homomorphism $\alpha$ of $S/\gamma$ onto $T$ such that

$$
\begin{array}{ccc}
W_B & \overset{\ell}{\leftarrow} & S \\
\downarrow & & \downarrow \alpha \\
S & \overset{\pi}{\rightarrow} & W_B/\gamma_1 \\
\downarrow & & \downarrow \gamma_1^h \\
S & \overset{\pi}{\rightarrow} & W_B
\end{array}
$$

(4)

commutes.

We know that the set of inverses of $(x, t)$ in $S$ is $V(x) \times \{t^{-1}\}$, where $V(x)$ is the set of inverses of the element $x$ in $W_B$. Hence, using the characterization of $\gamma$ by Proposition 2.2, we have that

$$
\gamma = \{(x, t), (y, u) : (x, t) \in S \times S : V(x) \times \{t^{-1}\} = V(y) \times \{u^{-1}\}\}
$$

$$
= \{(x, t), (y, u) : x = y \text{ and } \forall t \in S : V(x) = V(y)\}.
$$

On the other hand, $t = u$ implies $x\mathcal{T} = u\mathcal{T}$, which in turn implies $x\gamma_1^h = u\gamma_1^h$ since $(x, t), (y, u) \in S$. Thus, using the characterization of $\gamma_1$ by Proposition 2.2, we have that if $t = u$ then it follows that $V(x) = V(y)$. Therefore,

$$
\gamma = \{(x, t), (y, u) : (x, t) \in S \times S : t = u\}
$$

$$
= \pi \circ \pi^{-1}
$$

and so the mapping $\alpha : S/\gamma \rightarrow T$ in the diagram (4) is an isomorphism.

This completes the proof of Theorem.
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References

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Myongji University