A REMARK ON THE KRULL DIMENSION

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1. Introduction

Let $A$ be a commutative local ring and $\mathcal{C}_A$ the category of finite $A$-modules. Then $d: \mathcal{C}_A \to \mathbb{N}$ defined by $d(E)$, Krull dimension of $E$, satisfies the following properties:

1) $\dim(A/\mathfrak{m}) = 0$
2) If $0 \to E' \to E \to E'' \to 0$ is an exact sequence, then $d(E) = \max(d(E'), d(E''))$.
3) If $0 \to E \to E/aE \to 0$ where $a \in \mathfrak{m}$ is an exact sequence then $d(E) = -1 + d(E/aE)$.

A main purpose of this note is to show that the above three properties do characterize dimension, i.e., $d: \mathcal{C}_A \to \mathbb{N}$ with the above three properties is unique. For the sake of readers, we also give a proof of above properties for Krull dimension based on the notion of Hilbert-Samuel polynomial.

2. Definitions and preliminaries

Let $A$ be a noetherian local ring with maximal ideal $\mathfrak{m}$, $E$ a finite $A$-module, $(E_n)$ a stable $\mathfrak{m}$-filtration of $E$. Let

$\text{gr}(A) = \bigoplus_{r=0}^{\infty} A/\mathfrak{m}^r$, $\text{gr}(E) = \bigoplus_{r=0}^{\infty} A/\mathfrak{m}^rE$, $\text{gr}_r(A) = \mathfrak{m}^r/\mathfrak{m}^{r+1}$,

then $\text{gr}_0(A) = A/\mathfrak{m}$ is a field and hence $\text{gr}(A)$ is a noetherian ring, and $\text{gr}(E)$ is a finite $\text{gr}(A)$-module. $\text{gr}_r(E) = \mathfrak{m}^rE/\mathfrak{m}^{r+1}E$ is a noetherian $A$-module annihilated by $\mathfrak{m}$.

If $\{x_1, x_2, \cdots, x_p\}$ generates $\mathfrak{m}$, the image $\bar{x}_i$ of the $x_i \in \mathfrak{m}$ generate $\text{gr}(A)$ as an $A/\mathfrak{m}$-algebra and $\bar{x}_i$ has degree 1.

**Proposition 1.** $P_E(t) = \sum_{r=0}^{\infty} C_r t^r$, where $C_r = [\mathfrak{m}^rE/\mathfrak{m}^{r+1}E : A/\mathfrak{m}]$ is of the form $f(t)/(1-t)$ for some $f(t) \in \mathbb{Z}[t]$. 

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Proof. We shall prove by the induction on \( p \), the number of generators of \( \text{gr}(A) \) over \( A/\mathfrak{m} \). Let \( p = 0 \). Then \( \mathfrak{m}^n/\mathfrak{m}^{n+1} = 0 \) for all \( n > 0 \), so that \( \text{gr}(A) = A/\mathfrak{m} \) and \( \text{gr}(E) \) is a finitely generated \( A/\mathfrak{m} \)-vector space, and hence \( \mathfrak{m}^nE/\mathfrak{m}^{n+1}E = 0 \) for all \( n \gg 0 \). Thus \( P_E(t) \) is a polynomial.

Suppose \( p > 0 \) and the proposition true for \( p - 1 \). Multiplication by \( \alpha_p \) is an \( A \)-module homomorphism of \( \mathfrak{m}^nE/\mathfrak{m}^{n+1}E \) into \( \mathfrak{m}^{n+1}E/\mathfrak{m}^{n+2}E \) and hence it gives an exact sequence:

\[ 0 \rightarrow P_n/P_{n+1} \rightarrow \mathfrak{m}^nE/\mathfrak{m}^{n+1}E \rightarrow \mathfrak{m}^{n+1}E/\mathfrak{m}^{n+2}E \rightarrow Q_{n+1}/Q_{n+2} \rightarrow 0 \]

Let \( P = \bigoplus P_v/P_{v+1}, \ Q = \bigoplus Q_v/Q_{v+2} \). These are both finitely generated \( A \)-modules and both annihilated by \( \bar{x}_p \), hence they are \( A/\mathfrak{m} [\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_p] \)-module. Applying an additive function to \( *1 \) we get

\[ \lambda(P_n/P_{n+1}) = -\lambda(\mathfrak{m}^nE/\mathfrak{m}^{n+1}E) + \lambda(\mathfrak{m}E/\mathfrak{m}E) - \lambda(Q_{n+1}/Q_{n+2}) = 0. \]

Multiplying by \( \mathfrak{m}^{n+1} \) and summing with respect to \( n \) we get \( (1-t)P_E(t) = P_0(t) - tP_P(t) + h(t) \), where \( h(t) \) is a polynomial. By the induction assumption \( P_0(t) \) and \( P_P(t) \) are rational function of the form \( g(t)/(1-t) \), and hence

\[ (1-t)P_E(t) = P_0(t) - tP_P(t) + h(t) = f(t)/(1-t)^p. \]

Therefore, \( P_E(t) = f(t)/(1-t)^{p+1} \).

Corollary. For all \( n \gg 0 \), \( \lambda(\mathfrak{m}^nE/\mathfrak{m}^{n+1}E) \) is a polynomial in \( n \) of degree \( p - 1 \).

Proof. By the above proposition \( \lambda(\mathfrak{m}^nE/\mathfrak{m}^{n+1}E) \) is the coefficient of \( t^n \) in \( f(t)/(1-t)^p \). Suppose \( f(1) \neq 0 \) and \( f(t) = \sum_{k=0}^{N} a_k t^k \). Since

\[ (1-t)^{-p} = \sum_{k=0}^{N} (\binom{n+k-1}{k}) t^k, \quad \lambda(\mathfrak{m}^nE/\mathfrak{m}^{n+1}E) = \sum_{k=0}^{N} a_k (\binom{n+k-1}{k}) \text{ for all } n \geq N. \]

Therefore \( \mathfrak{d}(n) = \lambda(\mathfrak{m}^nE/\mathfrak{m}^{n+1}E) \) is a polynomial in \( n \) of degree \( \leq p - 1 \). It follows that the function

\[ g_E(n) = \mathfrak{d}(n) = \lambda(E/\mathfrak{m}E) = \sum_{i=0}^{n-1} \lambda(\mathfrak{m}^iE/m^{i+1}E) \]

is also a polynomial in \( n \) of degree \( \leq p \) for all \( n \gg 0 \). This \( g_E(n) \) is the Hilbert–Samuel polynomial in \( n \) of \( E \) with respect to \( \mathfrak{m} \). We shall let \( \deg g_E(n) = d(E) \).

Remark. Define \( \mathfrak{d}(E) \) to be the order of the pole at \( 1 \) in \( P_E(t) \). As we know easily \( \mathfrak{d}(E) = d(E) - 1 \). In fact,

\[ \lambda(E/\mathfrak{m}^{n+1}E) = \lambda(E/\mathfrak{m}E) + (\mathfrak{m}E/\mathfrak{m}^2E) + \cdots + (\mathfrak{m}^nE/\mathfrak{m}^{n+1}E). \]

Put \( P_E(t) = \sum_{v=0}^{\infty} C_v t^v \), where \( C_v = \lambda(\mathfrak{m}^vE/\mathfrak{m}^{v+1}E) \). Then since \( C_v = \lambda_v - \lambda_{v-1} \), where \( \lambda_v = C_1 + C_2 + \cdots + C_v \), we have
A remark on the Krull dimension

\[ P_E(t) = \sum_{\nu=0}^{m} (\lambda(E/\mathfrak{m}^{\nu+1}E) - \lambda(E/\mathfrak{m}^{\nu}E)) t^\nu \]

\[ = \sum \lambda(E/\mathfrak{m}^{\nu+1}E) t^\nu - (\sum (\lambda(E/\mathfrak{m}^{\nu}E) t^{\nu-1}) t = (1-t) \sum \lambda(E/\mathfrak{m}^{\nu+1}E) t^\nu. \]

Therefore,

\[ \sum \lambda(E/\mathfrak{m}^{\nu+1}E) t^\nu = \frac{1}{1-t} P_E(t) = \frac{1}{1-t} \sum \lambda(E/\mathfrak{m}^{\nu}E) t^\nu. \]

As we expect, \( \delta(E) = d(E) - 1. \)

**PROPOSITION 2.** Let \( A, \mathfrak{m}, E \) be as in Proposition 1 and \( \mathcal{O}_A \) the category of finite \( A \)-modules. For any objects \( E, E', E'' \in \mathcal{O}_A \), if

\[ 0 \to E' \to E \to E'' \to 0 \]

is an exact sequence of finite \( A \)-module, then \( d(E) = \max (d(E'), d(E'')) \)

**Proof.** From given exact sequence we know

\[ 0 \to E' \oplus \mathfrak{m}^n E'/\mathfrak{m}^n E \to E'/\mathfrak{m}^n E \to E'/\mathfrak{m}^n E'' \to 0 \]

is an exact sequence of finite \( A \)-module, and so \( E'/\mathfrak{m}^n E = E''/\mathfrak{m}^n E'' \).

Since

\[ \lambda(E''/\mathfrak{m}^n E'') = \lambda(E'/\mathfrak{m}^n E) \leq \lambda(E/\mathfrak{m}^n E), \]

we get \( d(E'') \leq d(E) \). Furthermore,

\[ \lambda_a^E(n) - \lambda_a^{E'}(n) = \lambda(E/\mathfrak{m}^n E) - \lambda(E'/\mathfrak{m}^n E') \]

\[ = \lambda(E'/\mathfrak{m}^n E') - \lambda(E'/\mathfrak{m}^n E) = \lambda(E'/\mathfrak{m}^n E'/\mathfrak{m}^n E) \]

\[ = \lambda(E'/\mathfrak{m}^n \cap \mathfrak{m}^n E), \]

and there exists \( r > 0 \) such that \( E'/\mathfrak{m}^n E \subset \mathfrak{m}^{n-r} E' \) for all \( n > r \) by Artin-Rees. Thus

\[ \lambda(E'/\mathfrak{m}^n E') \geq \lambda(E'/\mathfrak{m}^n \cap \mathfrak{m}^n E) \geq \lambda(E'/\mathfrak{m}^{n-r} E'). \]

This means that \( \lambda_a^E(n) - \lambda_a^{E'}(n) \) and \( \lambda_a^{E'}(n) \) have the same degree and the same leading term.

**PROPOSITION 3.** Let \( A, \mathfrak{m}, E \) be as in proposition 1, and \( a \in \mathfrak{m} \) non-zero divisor on \( E \). Then \( d(E) - 1 = d(E/aE) \).

**Proof.** From the given condition, we get an exact sequence:

\[ 0 \to aE \to E \to E/aE \to 0 \]

and hence

\[ 0 \to aE + \mathfrak{m}^n E/\mathfrak{m}^n E \to E/\mathfrak{m}^n E \to E/aE/\mathfrak{m}^n (E/aE) \to 0 \]

and then

\[ \lambda_a^{E/aE}(n) = \lambda(E/aE + \mathfrak{m}^n E) = \lambda(E/\mathfrak{m}^n E) - \lambda(aE + \mathfrak{m}^n E/\mathfrak{m}^n E), \]

\[ aE + \mathfrak{m}^n E/\mathfrak{m}^n E \cong aE/aE \cap \mathfrak{m}^n E \cong E/(\mathfrak{m}^n E:a) \text{ and } \mathfrak{m}^{n-1} E \subseteq (\mathfrak{m}^n E:a). \]

Hence

\[ \lambda_a^{E/aE}(n) \geq (E/\mathfrak{m}^n E) - \lambda(E/\mathfrak{m}^{n-1} E) = \lambda_a^E(n) - \lambda_a^E(n-1). \]
It follows that $d(E/aE) \geq d(E) - 1$. On the other hand, $aE \cong E$ as $A$-modules by the hypothesis on $a$. We have an exact sequence:

$$0 \to aE/aE \to E/M^nE \to E/aE \to 0.$$ 

Hence

$$\lambda(aE/aE \cap M^nE) = \lambda(E/M^nE) + \lambda(E/aE/M^n(E/aE)) = 0$$

for all $n \geq 0$. By the Artin–Rees, $aE \cap M^nE$ is a stable $M$-filtration of $E$.

Since $aE \cong E$, $\lambda(E/aE \cap M^nE)$ and $\lambda_n(a)$ have the same leading term because the degree and leading coefficient of Hilbert–Samuel polynomial depend only on $E$ and $m$, not on the filtration chosen.

Therefore $d(E/aE) \leq d(E) - 1$.

### 3. Main theorem

**Theorem.** Let $A$ be a local noetherian ring with maximal ideal $m$, $E$ a finite $A$-module, $gr(E) = \bigoplus_{p=0}^{\infty} M^nE/M^{n+1}E$, $P_E(t) = \sum_{p=0}^{\infty} C_p t^p$, where $C_p = [M^nE/M^{n+1}E: A/m]$, $\alpha(E)$ the order of the pole at 1 in $P_E(t)$ and $\Theta_A$ the category of finite $A$-modules.

Define $E \to \lambda(E) \in N$ non negative, then following hold:

1. $\lambda^*E = 0$ for some $n > 0$ $\implies$ (1') $\alpha(E) = \alpha(A/m) = 0$
2. $0 \to E' \to E \to E'' \to 0$ exact and $\alpha(E) = \max(\alpha(E'), \alpha(E''))$
3. $0 \to aE \to E \to E/aE \to 0$ exact, where $a \in M$

$$\implies \alpha(E/aE) = \alpha(E) - 1.$$ 

Conversely, this map is uniquely determined by the above conditions.

**Proof.** (1) $M^nE = 0$ for some $n > 0$ implies $M(M^nE) = 0$, and hence $M^{n+1}E = 0$.

Since $M^{n+k}E = 0$ for all $k \geq 0$, $M^{n+k}/M^{n+k+1}E = 0$.

Thus $C_{n+1} = \dim_{A/m}(M^{n+1}E/M^{n+2}E) = 0$. Therefore, $P_E(t) = \sum_{p=0}^{\infty} C_p t^p = \sum_{p=0}^{\infty} C_p t^p$

because $C_p = 0$ if $p > n$, so the order of the pole is zero, i.e., $\alpha(E) = 0$.

(1) $\iff$ (1') From the given condition, we get a chain $E \supset E_1 \supset \cdots \supset E_r = 0$ of submodules such that $E_i/E_{i+1} = A/P_i$ by J.P. Serre.

Since $P_i \supset M \implies P_i \supset M$, $P_i = M$. Hence $\alpha(E) = \alpha(A/P) = 0$.

(2) & (3) follows from Proposition 2, 3 and Remark.

Conversely, by J.P. Serre, there exists a chain such that $E = E_0 \supset E_1 \supset \cdots \supset E_r = 0$, where $E_i/E_{i+1} = A/P_i$ for some $i$.

From

$$0 \to E_1 \to E \oplus E_1 \to 0,$$

$$0 \to E_2 \to E_1 \to E_1/2E_2 \to 0,$$

$$\cdots$$
we know
\[ d(E) = \max \{ d(E/E_1), \ldots, d(E/E_r) \} \]
\[ = \max \{ d(A/P_1), \ldots, d(A/P_r) \}. \]
If \( d(A/P) = \dim(A/P) \), then \( d(E) = \dim(E) \). Thus if \( d(E) \neq \dim(E) \) for some module \( E \) then \( d(A/P) \neq \dim(A/P) \) for some prime ideal \( P \).

Assume that \( d(E) \neq \dim(E) \), and then choose a maximal one among all prime ideals \( P \) for which \( d(A/P) \neq \dim(A/P) \).

Let \( P \) be a maximal one.

(a) \( P \neq \% \) because if \( P = \% \) then \( d(A/P) = 0 \) by (1') whereas \( \dim(A/\%) = 0 \).

(b) Since \( P \subseteq \% \) we can pick \( a \in \% - P \). The multiplication by \( a \) in \( A/P \) is one-one, i.e., \( 0 \to A/P \to A/P \to A/(P+aA) \to 0 \) is exact. Then, by (3), we have
\[ d(A/P+aA) = d(A/P) - 1, \text{ i.e., } d(A/P) = 1 + d(A/P+aA). \]
However choose \( A/P+aA = E \supseteq E_1 \supseteq \ldots \supseteq E_r = 0 \) such that \( E_i/E_{i+1} \cong A/P_i \), then \( P_i \supseteq P+aA \supseteq P \). Because \( P \) was a maximal amongst \( d(A/P) \neq \dim(A/P) \) we must have \( d(A/P_i) = \dim(A/P_i) \) for all \( i \). Therefore,
\[ \max(d(E/E_1), d(E/E_2), \ldots) = d(A/P+aA) \]
\[ = \max(d(A/P_1), d(A/P_2), d(A/P_3), \ldots) = \dim(A/P+aA). \]
Hence
\[ d(A/P) = 1 + d(A/P+aA) = 1 + d(A/P+aA) = \dim(A/P), \]
which is a contradiction.

**Corollary** Let \( A \to B \) be a local map of noethorian local rings, and \( E \) a finite \( B \)-module which is \( A \)-flat. Then for any finite \( A \)-module \( M \) we have
\[ \dim_A(M) = \dim_B(M \otimes_A E) - \dim_B(E/\%E). \]

**Proof.** Let \( \Theta_A \to N \), where \( \delta(M) = \dim_B(M \otimes_A E) - \dim_B(E/\%E) \). Then
(i) \( \delta(A/\%) = \dim_B(A \otimes E) - \dim_B(E/\%E) = \dim_B(E/\%E) - \dim_B(E/\%E) = 0. \)
(ii) \( 0 \to M' \to M \to M'' \to 0 \) is exact and
\[ 0 \to M' \otimes_A E \to M \otimes_A E \to M'' \otimes_A E \to 0 \] is exact since \( E \) is \( A \)-flat.
So \( \dim_B(M \otimes_A E) = \max(\dim_B(M' \otimes A E), \dim_B(M'' \otimes A E)). \) Therefore,
\[ \delta(M) = \max(\delta(M'), \delta(M'')). \]
(iii) \( 0 \to M \to M \to M/aM \to 0, \) where \( a \in \% \), is exact.
\[ \Rightarrow 0 \to M \otimes_A E \to M \otimes_A E \to M/aM \otimes_A E \to 0 \] is exact.
\[ \Rightarrow 0 \to M \otimes_A E \to M \otimes_A E \to M \otimes_A E/a(M \otimes_A E) \to 0 \] is exact.
\[ \Rightarrow \dim_B(M \otimes_A E) = 1 + \dim_B(M \otimes_A E/a(M \otimes_A E)). \]
So \( \delta(M) = 1 + (M/aM). \)
By uniqueness,
\[ \dim_A(M) = \dim_B(M \otimes_A E) - \dim_B(E/\%E). \]
References


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