FINITELY GENERATED CONVERGENCE SPACES

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1. Introduction

In this note we introduce the concept of finitely generated convergence spaces, and then find some internal characterizations of those spaces. Also it is shown that in the category $\mathbf{Cv}$ of convergence spaces and continuous maps, the full subcategory $\mathbf{FCv}$ of $\mathbf{Cv}$ formed by all finitely generated convergence spaces is the bicoreflective hull of the class of all finite convergence spaces and that as $\mathbf{Cv}$, $\mathbf{FCv}$ is also cartesian closed. It is known [7] that the full subcategory $\mathbf{FTop}$ of $\mathbf{Top}$ formed by finitely generated topological spaces contains all indiscrete spaces. However we show that this is not the case for $\mathbf{FCv}$. Finally, we give a characterization of objects of $\mathbf{FCv} \cap \mathbf{Top}$.

All categorical concepts will be used in the sense of Herrlich [5, 6, 7].

2. Finitely Generated Convergence Spaces

For any set $X$, let $P(X)$ and $F(X)$ denote the power set of $X$ and the set of all filters on $X$, respectively.

The following definition is due to Fischer [4].

DEFINITION 1. Let $X$ be a set and $c : X \to P(F(X))$ a map. The map $c$ is called a convergence structure on $X$ if it satisfies the following:

C1) for any $x \in X$, $\emptyset \in c(x)$.
C2) if $\mathcal{I} \in c(x)$ and $\mathcal{I} \subseteq \mathcal{Q}$, then $\mathcal{Q} \in c(x)$.
C3) if $\mathcal{I}, \mathcal{Q} \in c(x)$, then $\mathcal{I} \cap \mathcal{Q} \in c(x)$.

In this case, $(X, c)$ is called a convergence space.

REMARK. In [4], convergence spaces have been called limit spaces.

NOTATION. Let $(X, c)$ be a convergence space. If $\mathcal{I} \subseteq c(X)$, then $x$ is called a limit of $\mathcal{I}$, or $\mathcal{I}$ is said to converge to $x$, and we write $\mathcal{I} \overset{c}{\to} x$ or $\mathcal{I} \to x$, when there is no confusion about the convergence structure $c$.

A filter base in a convergence space is also said to converge to $x$ if the filter generated by the filter base converges to $x$. 
DEFINITION 2. Let \((X, c)\) and \((Y, c')\) be convergence spaces and \(f : X \to Y\) a map. Then \(f\) is said to be \textit{continuous} on \((X, c)\) to \((Y, c')\) if for any filter \(\mathcal{F} \subseteq c(x)\), \(f(\mathcal{F}) \subseteq c'(f(x))\).

It is clear that the class of all convergence spaces and continuous maps between them forms a category, which will be denoted by \(Cv\). Moreover, \(Cv\) is a properly fibred cartesian closed topological category (see [2, 6, 9]).

Now we are ready to introduce the concept of finitely generated convergence spaces.

DEFINITION 3. A convergence space \((X, c)\) is said to be \textit{finitely generated} if there is a final sink \((f_i : F_i \to X)_{i \in I}\) such that each \(F_i\) is a finite convergence space.

**Lemma.** Let \((X, c)\) be a convergence space and \(\mathcal{F}\) a filter on \(X\). Then the following are equivalent:

a) There is a finite family \(\{F_1, \ldots, F_n\}\) of finite subspaces of \(X\) and for each \(k = 1, \ldots, n\), there is a filter \(\mathcal{F}_k\) on \(F_k\) converging to \(x\) in \(F_k\), and hence in \(X\) such that \(\bigcap j_{F_k}(\mathcal{F}_k) = \bigcap \mathcal{F}_k \subseteq \mathcal{F}\), where \(j_{F_k}\) denotes the natural embedding of \(F_k\) into \(X\).

b) There is a finite subspace \(F\) of \(X\) such that \(x \in F \in \mathcal{F}\) and \(\mathcal{F}|_F = \{G | G \subseteq F \text{ and } G \in \mathcal{F}\} \to x\).

c) There is a finite subspace \(F\) of \(X\) such that \(F \in \mathcal{F}\) and the principal filter \([F]\) generated by \(F\) converges to \(x\).

**Proof.** a) \(\Rightarrow\) b). Let \(F = UF_k\). Then for each \(k\), \(F \supseteq F_k\) and hence \(F \in \bigcap \mathcal{F}_k\). Thus \(F \in \mathcal{F}\), and \(\bigcap \mathcal{F}_k \subseteq \mathcal{F}|_F\). Therefore \(\mathcal{F}|_F\) converges to \(x\) by C2 and C3.

b) \(\Rightarrow\) a). It is trivial.

b) \(\Rightarrow\) c). Since \(F\) is finite, \(\bigcap \{G | G \subseteq F \text{ and } G \in \mathcal{F}\} = K\) is a member of \(\mathcal{F}\) and \([K] = \mathcal{F}|_F\) converges to \(x\).

c) \(\Rightarrow\) b). Since \(F \in \mathcal{F}\), \(\mathcal{F}|_F\) contains \([F]\). Thus \(\mathcal{F}|_F \to x\).

**Theorem 1.** For a convergence space \(X\), the following are equivalent:

1) \(X\) is finitely generated.

2) The sink \(\{j_F : F \to X | F\text{ is a finite subspace of } X \text{ and } j_F \text{ is the natural embedding}\}\) is final.

3) For a filter \(\mathcal{F}\) on \(X\), \(\mathcal{F}\) converges to \(x\) iff either \(\mathcal{F} = \mathcal{F}_x\) or there is a finite subset \(F\) of \(X\) such that \(F \in \mathcal{F}\) and \([F]\) converges to \(x\).

4) For a filter \(\mathcal{F}\) on \(X\), \(\mathcal{F}\) converges to \(x\) iff either \(\mathcal{F} = \mathcal{F}_x\) or there is a finite subset \(\{x_1, \ldots, x_n\}\) of \(F\) such that each \(x_k (1 \leq k \leq n)\) converges to \(x\) and \(\bigcap \mathcal{F}_k \subseteq \mathcal{F}\).
Proof. 1) \iff 2). Suppose \( X \) is finitely generated, then there is a final sink \( (f_i : K_i \to X)_{i \in I} \) such that each \( K_i \) \((i \in I)\) is a finite convergence space. For each \( i \in I \) let
\[
K_i \xrightarrow{f_i} X = K_i \xrightarrow{h_i} f_i(K_i) \xrightarrow{j_i} X
\]
be the canonical factorization, i.e., \( j_i \) is the natural embedding and \( h_i(x) = f_i(x) \) \((x \in K_i)\). Since \( f_i(K_i) \) is also finite, and \( (j_i : f_i(K_i) \to X)_{i \in I} \) is again final, the sink \( \{j_F | F \text{ is a finite subspace of } X\} \) is final, because it contains the sink \( (j_i)_{i \in I} \).

The converse is immediate.

2) \iff 3). By the characterization of final sinks in \( \text{Cv} \) (see \([2,9]\)), the sink \( \{j_F : F \to X | F \text{ is a finite subspace of } X\} \) is final iff for a filter \( \mathcal{F} \) on \( X \) to converge to \( x \) in \( X \) it is necessary and sufficient that either \( \mathcal{F} = \{x\} \) or \( \mathcal{F} \) satisfies a) of the above lemma. Hence using the above lemma, we have the equivalence.

3) \iff 4). It follows from the fact that for any finite subset \( F \) of \( X \), \([F]\) coincides with \( \cap \{x \mid x \in F\} \).

**NOTATION.** The full subcategory of \( \text{Cv} \) formed by all finitely generated convergence spaces will be denoted by \( \text{FCv} \).

**Theorem 2.** The category \( \text{FCv} \) is bicoreflective in \( \text{Cv} \) and \( \text{FCv} \) is the bicoreflective hull in \( \text{Cv} \) of the class of all finite convergence spaces.

**Proof.** By the above theorem and the fact that the composition of final sinks is again final, we can conclude that \( \text{FCv} \) is closed under the formation of final sinks in \( \text{Cv} \). Hence \( \text{FCv} \) is bicoreflective in \( \text{Cv}[6] \). More precisely, let us find the coreflection of any convergence space \((X, c)\). We define \( c_f : X \to P(F(X)) \) as follows: for any \( x \in X \), \( c_f(x) = \{\mathcal{F} | \text{there is a finite subset } \{x_1, \ldots, x_n\} \text{ of } X \text{ such that } x_i \text{ converges to } x \text{ in } (X, c) \ (1 \leq k \leq n) \text{ and } \cap x_i \subseteq \mathcal{F} \} \). Then it is immediate that \( c_f \) is a convergence structure on \( X \) and that the identity map \( 1_X : (X, c_f) \to (X, c) \) is continuous. It remains to show that for any continuous map \( f : (Y, c') \to (X, c) \) with \( (Y, c') \in \text{FCv} \), \( f : (Y, c') \to (X, c_f) \) is also continuous. Suppose \( \mathcal{F} \to y \) in \((Y, c')\), then there is a finite subset \( \{y_1, \ldots, y_n\} \) such that \( y_k \to y \) in \((Y, c') \ (1 \leq k \leq n) \), and \( \cap y_k \subseteq \mathcal{F} \). Since \( f(\mathcal{F}) = f(\cap y_k) \to f(y) \) in \((X, c) \ (1 \leq k \leq n) \), and \( f(\cap y_k) = \cap f(y_k) \subseteq f(\mathcal{F}) \), \( f(\mathcal{F}) \) converges to \( f(y) \) in \((X, c_f)\). Thus \( f : (Y, c') \to (X, c_f) \) is continuous.

The second part is immediate from the above theorem and the results in \([6,7]\).

Since \( \text{Cv} \) is a properly fibred topological category, the following is immediate from the above theorem.
Corollary. The category $\text{FCv}$ is a properly fibred topological category and closed under the formation of coproducts and quotients in $\text{Cv}$.

Proposition 3. The category $\text{FCv}$ is closed under the formation of subspaces and finite products in $\text{Cv}$.

Proof. Let $(X,c)$ be a finitely generated convergence space and $A$ a subset of $X$. If a filter $\mathcal{F}$ on $A$ converges to $x$ in the subspace $(A,c_A)$, then $\mathcal{F} \to x$ in $(X,c)$. Hence there is a finite subset $F$ of $X$ such that $F \subseteq A$, $F \in \mathcal{F}$ and $[F] \to x$ in $(A,c_A)$. Since $F \subseteq A$, $F \in \mathcal{F}$ and $[F] \to x$ in $(A,c_A)$.

The empty product, i.e. the singleton space obviously belongs to $\text{FCv}$. Let $(X,c), (Y,d) \in \text{FCv}$. If a filter $\mathcal{F}$ on $X \times Y$ converges to $(x,y)$ in $X \times Y$, then there is a filter $\mathcal{F}_1$ on $X$ and a filter $\mathcal{F}_2$ on $Y$ such that $\mathcal{F}_1 \to x$, $\mathcal{F}_2 \to y$ and $\mathcal{F}_1 \times \mathcal{F}_2 \subseteq \mathcal{F}$. Since $(X,c)$ and $(Y,d)$ belong to $\text{FCv}$, there is a finite subset $F$ of $X$ and a finite subset $G$ of $Y$ such that $F \in \mathcal{F}_1$, $G \in \mathcal{F}_2$, $[F] \to x$, and $[G] \to y$. Therefore $F \times G$ is finite, $F \times G \in \mathcal{F}_1 \times \mathcal{F}_2 \subseteq \mathcal{F}$ and $[F \times G] \to (x,y)$.

Using the above proposition together with the result in [1] and the fact that $\text{Cv}$ is cartesian closed, we have the following:

Theorem 4. The category $\text{FCv}$ is cartesian closed.

We note that the full subcategory $\text{FTop}$ of $\text{Top}$ formed by finitely generated topological spaces contains all indiscrete spaces [7].

Remark. There is an indiscrete space which does not belong to $\text{FCv}$. For example, let $(X,c)$ be an infinite indiscrete space and $\mathcal{F} = \{X\}$. Then $\mathcal{F} \to x \in X$. But there is no finite subset $F$ of $X$ such that $[F] \subseteq \mathcal{F}$. Hence $(X,c)$ is not finitely generated.

Proposition 5. A topological convergence space $(X,c)$ belongs to $\text{FCv}$ iff for each $x \in X$, there is a finite neighborhood $F_x$ of $x$ such that $\{F_x\}$ is a local base at $x$.

Proof. Since $(X,c)$ is topological, a filter $\mathcal{F}$ on $(X,c)$ converges to $x$ iff it contains the neighborhood filter $\mathcal{U}_x$ of $x$. Hence $(X,c)$ belongs to $\text{FCv}$ iff there is a finite neighborhood $F_x$ of $x$ such that $[F_x] \to x$, i.e. $\mathcal{U}_x \subseteq [F_x] \subseteq \mathcal{U}_x$. Therefore $(X,c)$ belongs to $\text{FCv}$ iff $\{F_x\}$ is a local base at $x$ and $F_x$ is finite.

References


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