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Completely Indecomposable Modules over a Ring.

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In this paper we always assume that all rings have an identity and every module is unitary. For rings A and B we shall write ${}_AM(M_A)$ to denote a left (right) A-module M and ${}_AM_B$ to denote a two sided (A, B)-module.

The main purpose of this paper is to find necessary and sufficient conditions for a module to be a direct sum of completely indecomposable module over a noncommutative ring.

Definition 1.

•A module ${}_AM$ is termed uniform if $N_1 \cap N_2 \neq 0$ for every pair of nonzero submodules N_1 and N_2. A ring A is right (left) uniform if it is uniform as a right (left) module over itself.

Definition 2.

A left A-module M is called completely indecomposable if the following conditions are satisfied;

(i) ${}_AM$ is uniform, Artinian and Noetherian,

(ii) There exists another ring B such that M is a two sided (A, B)-module and such that M_B is uniform, Artinian and Noetherian.

A ring is called completely indecomposable, if it is left and right Artinian and uniform.

For ${}_AM$, the annihilator of a set $X \subseteq M$ is written as $\mathrm{Ann}(X) = \{a \in A \mid ax = 0,$ for all $x \in X\}$. For a set $Z \subseteq A$, the left annihilator of Z in A is denoted by $l(Z) = \{a \in A \mid az = 0,$ for all $z \in Z\}$.

If ${}_AM$ is Artinian and Noetherian, then M is a direct sum of indecomposable submodules, $M = M_1 \oplus M_2 \oplus M_3 \oplus \cdots \oplus M_n$, where M_i is an indecomposable submodule for $i = 1, 2, 3, \cdots, n$. If $M = M_1' \oplus M_2' \oplus M_3' \oplus \cdots \oplus M_m'$ is another decomposition of M into a direct sum of indecomposable submodules, after suitable ordering, $m=n$ and M_i is A-isomorpic to M_i' for $i=1, 2, 3, \cdots n$, according to the Krull-Schmidt theorem ([1], p. 277). Consequently, the annihilating ideals of $M_1, M_2 \cdots, M_n$ are completely determined by M and do not depend on the particular decomposition of M which is chosen.

Definition 3.

Let ${}_AM$ be an Artinian and Noetherian and let $M=M_1 \oplus M_2 \oplus M_3 \oplus \cdots \oplus M_n$ be a decomposition of M into a direct sum of indecomposable submodules. Then the annihilating ideals $Q_1, Q_2, Q_3, \cdots, Q_n$ respectively of $M_1, M_2, M_3, \cdots, M_n$ are called the elementary divisor ideals.

Definition 4.

An ideal Q of a ring is right (left) intersection irreducible if Q is not the intersection of two right (left) ideals that properly contain Q.

Definition 5.

A two sided (A, B)-module M is termed cyclic provided $M=Ax=xB$, for $x \in M$.

Lemma 1.

A completely indecomposable module is cyclic if and only if its annihilating ideal is intersection irreducible.

(**Proof**) Let ${}_AM$ be a completely indecomposable module and Q as annihilating ideal. If ${}_AM$ is cyclic, M is A-isomorpic to A/Q which implies that then A/Q is completely indecomposable and hence Q is intersection irreducible ([8], p. 127). Conversly, if Q is intersection irreducible, A/Q is completely indecomposable ([8], p. 127). Consequently, M and A/Q are then two completely indecomposable modules with the same annihilating ideal Q in A. Thus by Theorem 5. 1 ([8]) we have M is A-isomorphic to A/Q and hence then M is cyclic since A/Q is cyclic.

Definition 6.

An ideal Q of a ring A is primary if a, $b \in A$, and $ab \in Q$, then $a \notin Q$ implies $b^m \in Q$; $b \notin Q$ implies $a^n \in Q$ for positive integers m, n.

Theorem 1.

Let Q be an ideal of a ring A, where A/Q is left Artinian, then A/Q is a local ring if Q is primary.

(**Proof**) If A/Q is local, then obviously Q is primary since the non-units of A/Q form a nilpotent ideal. If Q is primary, then as in ([4], p. 80), the nilpotent elements from an ideal in A/Q, and non-nilpotent elements are regular in A/Q Since A/Q is left Artinian, the regular elements are units.

Definition 7.

A module ${}_AM$ is called semi-completely indecomposable if M is the direct sum of finite number of completely indecomposable A-modules.

In [5], Feller showed that the following theorems.

Cyclic Decomposition Theorem.

A semi-completely indecomposable module ${}_AM$ is a direct sum of cyclic completely indecomposable modules M_i, i= 1, 2, 3, …, n, where M_i is A-isomorphic to A/Q_i and where Q_i is the elementary divisor ideal, if and only if ${}_AM$ is Artinian, Noetherian, and Q_i is right and left intersection irreducible and A/Q_i is right Artinian, for $i=$ 1, 2, 3, …, n.

Corollary 1.

Let the ring A be left and right Artinian and M a finitely generated injective A-moduble. If the elementary divisor ideals are left and right intersection irreducible, then M is the direct sum of cyclic completely indecomposable modules as given theorem.

Lemma 2.

If A is right Artinian then any right A-module is Noetherian if and only if it is Artinian.

(**Proof**) Let N be a radical of A, then $N^p=O$ for some positive integer p, since the radical of a right Artinian ring is nilpotent [7]. Now we consider any Artinian right A-moduble M. This has a chain of submodules

$$M\supset MN\supset MN^2\cdots\supset MN^p=O$$

with the factor modules $F_k=MN^{k-1}/MN^k$ $k=1,2,3,\cdots p$. Now F_k is annihilated by N, hence may be regarded as an A/N-module. By proposition 2 [7], F_k is Noetherian. Thus $MN^{p-1}(=F_p)$ and $MN^{p-2}/MN^{p-1}(=F_{p-1})$ are Noetherian, hence so is MN^{p-2} Continuing in this fashion, we see that M is Noetherian.

At this point we shall discuss the right Noetherian ring A. Harada showed that

A is right Noetherian, if every right A-injective module is a direct sum of completely indecomposable modules. [6]

From Cyclic decomposition theorem, Lemma 2, and the result of Harada, we can obtain following Theorem 2.

Theorem 2.

Let the ring A be left and right Noetherian and M a finitely generated injective A-module. M is the direct sum of cyclic completely indecomposable modules if and only if the elementary divisor ideals Q_i are left and right intersection irreducible and A/Q_i is right Noetherian for $i=1, 2, \cdots, n$.

Example.

Let R be a completely indecomposable ring and R_2 be the set of 2×2 matrices with elements in R. Let E_{ij} be the matrix with 1 in the (i, j)-position and zero elsewhere. Then $M=E_{11}R_2$ is Artinian and Noetherian as a right modul over R_2. Since the R_2 submodules of M are of the form $E_{11}I+E_{12}I$, and where I is a right ideal of R, then M is uniform over R_2. The $\mathrm{Hom}_R(M, M)=E_{11}R_2E_{11}=R$.[7] Then M is a left R-module which is Artinian and Noetherian but not uniform over R, since $RE_{11}\oplus RE_{12}=M$. This shows that in Definition 3, the condition which M is uniform over B is not superfluous. Let us focus on ${}_RM$. It is the direct sum of RE_{11} and RE_{12}, which are free (and cyclic) modules isomorpic as left R-module to R. Now $\mathrm{Hom}_R(RE_{1i}, RE_{1i})=R$, for $i=1, 2$. Thus ${}_RM$ is a semi-completely indecomposable moduble, which is isomorpic to the direct sum of the completely indecomposable modules RE_{11} and RE_{12}.

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Abstract

本 論文에서는 Noetherian Ring 上의 finitely generated injective module이 completely indecomposable modules의 direct sum으로 表示될 必要充分條件을 求하였다.