REGULAR GENERAL CONTACT MANIFOLDS

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1. Introduction

It has proved that a compact connected manifold $M^{2n+s}$ with a regular normal $f$-structure is the bundle space a principal $T^s$-bundle over a complex manifold $N^{2n}$. Moreover, if $M^{2n+s}$ is a $K$-manifold, then $N^{2n}$ is a Kähler manifold, [2]. In this work we prove that (Theorem 4.1) if the $K$-structure on $M^{2n+s}$ is an $S$-structure, then $N^{2n}$ is a Hodge manifold. Conversely (Theorem 4.4), given a Hodge manifold $N^{2n}$ and any $s>1$, there exists a principal toroidal bundle $M(N, T^s)$ over $N$, whose bundle space $M^{2n+s}$ has a regular $S$-structure.

2. Normal $f$-structures

A $C^\infty$-manifold $M^{2n+s}$, $n\geq 1$, is said to have an $f$-structure, if the structural group of its tangent bundle is reducible to $U(n)\times O(s)$. This is equivalent to the existence of a tensor field on $M$ of type $(1, 1)$, rank $2n$, satisfying $f^3+f=0$. Almost complex structures ($s=0$) and almost contact structures ($s=1$) are two examples of $f$-structures. If there exist vector fields $E_i$ and $1$-forms, $\eta^i$, $1\leq i\leq s$ such that

$$f(E_i)=0, \quad \eta^i(E_i)=\delta^i_j, \quad \eta^i \circ f=0, \quad f^2=-I+\sum_{i=1}^s \eta^i \otimes E_i$$

we say that $M^{2n+s}$ has a framed $f$-structure, or, simply an $(f, E_i, \eta^i)$-structure. A framed $f$-structure is normal if

$$S=[f, f]+\sum_{i=1}^s d\eta^i \otimes E_i$$

vanishes, where $[f, f]$ is the Nijenhuis tensor of $g$. In this case we have [3]:

1) $L_{E_i} \eta^j=0, \quad 2) \ [E_i, E_j]=0, \quad 3) \ L_{E_i} f=0, \quad 4) \ d\eta^i(fX, Y)=-d\eta^i(X, fY)$.

The equality 2) implies that the vertical distribution (the one generated by all the $E_i$) is integrable.

It is known that for any $(f, E_i, \eta^i)$-structure there exists a Riemannian metric $g$ which satisfies
g(X, Y) = g(fX, fY) + \sum_{i=1}^{s} \eta^i(X)\eta^i(Y).

A framed f-structure together with this metric is called a framed metric f-structure, or, simple, an, (f, E, \eta^i, g)-structure. The 2-form

\[ F(X, Y) = g(X, fY) \]

is called the fundamental 2-form of the (f, E, \eta^i, g)-structure. A K-structure is a normal (f, E, \eta^i, g)-structure whose fundamental 2-form is closed.

Let D be an integrable distribution of dimension h on a manifold N. A cubical coordinate neighborhood \((U, (x^1, \ldots, x^m))\) on N is said to be regular with respect to D if \(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^h}\) is a basis for D(\(p\)), for every \(p \in U\), and if each leaf of D intersects U in at most one n-dimensional slice of \((U, (x', \ldots, x^m))\). We call D regular if each point \(p \in N\) has a cubical coordinate neighborhood which is regular with respect to D.

An (f, E, \eta^i)-structure is said to be regular if the vertical distribution is integrable and regular, and if each \(E_i\) is regular (the distribution generated by \(E_i\) is regular).

Let's state the theorem mentioned at the beginning:

**THEOREM 2.1** (Blair, Ludden, Yano). Let \(M^{2n+s}\), \(n > 1\), be a compact connected manifold with a regular framed f-structure. Then \(M^{2n+s}\) is the bundle space of a principal toroidal bundle over a complex manifold \(N^{2n}\). Moreover, if the framed f-structure is a K-structure, then \(N^{2n}\) is a Kaehler manifold.

3. Toroidal bundles

Let \(T^1 = S^1\) and \(T^s = S^1 \times \cdots \times S^1\) be the one-dimensional and s-dimensional torus respectively. Since these Lie groups are commutative, by choosing \(A\), a nonzero element of the Lie algebra \(L(T^1)\) of \(T^1\), we identify \(L(T^1)\) with \(R\), and \(L(T^s) = L(T^1) \times \cdots \times L(T^1)\) with \(R^s\) by means of

\[ (0, \ldots, A, 0, \ldots, 0) \mapsto e_i, \]

where \(e_1, \ldots, e_s\) is the canonical basis of \(R^s\).

Let \(P[N, T^1]\) be the set of all \(T^1\)-bundles over the manifold \(N\). If \(P(N, T^1, \pi)\) and \(Q(N, T^s, \pi)\) are two elements in this set, on
we define the equivalent relation:

\[(u_1, v_1) \sim (u_2, v_2) \iff \exists t \in T^s \text{ such that } (u_1t, v_1t^{-1}) = (u_2, v_2).\]

The action of \(T^s\) on \(\Delta(P \times Q)\) given by \(((u, v), t) \mapsto (ut, v)\), induces an action of \(T^s\) on

\[P + Q = \Delta(P \times Q)\]

obtaining, in this way, the new \(T^s\)-bundle \(P + Q\). It is known that \(P[N, T^s]\) with this operation, "+", is an abelian group whose identity element is the trivial bundle \(N \times T^s\), [4].

If \(\omega\) is a connection form with curvature form \(\Omega\) of a bundle \(P(N, T^s)\), then

\[\omega = \sum_{i=1}^{s} \omega_i \otimes e_i \quad \text{and} \quad \Omega = \sum_{i=1}^{s} d\omega_i \otimes e_i.\]

Each real 2-form \(d\omega_i\) is horizontal and right invariant, therefore there exists a unique real 2-form \(\Omega_i^*\) on \(N\) satisfying \(d\omega_i = \pi^* \Omega_i^*\). Since the forms \(\Omega_i^*\) are closed, they determine \(s\) cohomology classes \([\Omega_i^*]_1, \ldots, [\Omega_i^*]_s\) in \(H^2(N, R)\). These cohomology classes are independent from the connection. In this way we get the function

\[\Psi: P[N, T^s] \to \bigoplus_{i=1}^{s} H^2(N, R) \text{ given by } P \mapsto ([\Omega_1^*], \ldots, [\Omega_s^*]).\]

Our intention now is to show that \(\Psi\) is a group homomorphism.

Suppose that \(\{\phi_{\beta \alpha}\}\) are the transition function of \(P(N, T^s)\) corresponding to some covering \(\{U_\alpha\}\). Each function \(\phi_{\beta \alpha} : U_\beta \cap U_\alpha \to T^s\) can be written as

\[(\phi_{\beta \alpha}^1, \ldots, \phi_{\beta \alpha}^s).\]

Now \(\{\phi_{\beta \alpha}^i\}\) are the transition functions of a 1-dimensional toroidal bundle \(P_i\) over \(N\). If we construct the whitney sum \(P_1 \oplus \cdots \oplus P_s\), it happens that a set of transition functions of this sum is precisely \(\{\phi_{\beta \alpha}\}\). In other words, \(P\) and \(P_1 \oplus \cdots \oplus P_s\) have the same transition function. Therefore we may assume that

\[P = P_1 \oplus \cdots \oplus P_s \quad \text{and} \quad P[N, T^s] = \bigoplus_{i=1}^{s} P[N, T^1].\]

Let \(h_i\) be the projection \(h_i : P_1 \oplus \cdots \oplus P_s \to P_i\). If \(\Omega_i\) is a curvature form on \(P_i\), there is a connection on \(P\) whose curvature form \(\Omega\) satisfies:

\[\Omega = \sum_{i=1}^{s} h_i^* \Omega_i \otimes e_i.\]
Therefore we can assume that the function
\[ \Psi : P[N, T^1] = \bigoplus_{i=1}^{s} P[N, T^1] \rightarrow \bigoplus_{i=1}^{s} H^2(N, R) \]
is given by \( \Psi = \Psi_1 \times \cdots \times \Psi_s \) where \( \Psi \) is the function
\[ \Psi : P[N, T^1] \rightarrow H^2(N, R) \]
such that \( \Psi(P_i) = [\Omega_i^*] \).

But this \( \Psi \) is precisely the function defined by S. Kobayashi in page 32 of [4]. Furthermore, he proves that \( \Psi : P[N, T^1] \rightarrow H^2(N, R) \) is a group homomorphism which sends \( P(N, T^1) \) onto \( H^2(N, Z)_b \) where \( H^2(N, Z)_b \) is the subgroup of \( H^2(N, R) \) formed by all the elements which contain an integral closed from.

Therefore

**Theorem 3.1.** The function
\[ \Psi : P[N, T^1] \rightarrow \bigoplus_{i=1}^{s} H^2(N, R) \]
\[ P \rightarrow ([\Omega_1^*], \ldots, [\Omega_s^*]) \]
is a group homomorphism, which sends \( P[N, T^1] \) onto
\[ \bigoplus_{i=1}^{s} H^2(N, Z)_b. \]

4. Regular S-structures

**Definition.** A manifold \( M^{2n+s} \) is said to have an s-contact structure if there exist on \( M \) s global, linearly independent 1-forms \( \eta_1, \ldots, \eta_s \) such that \( d\eta_1 = \cdots = d\eta_s \) has rank \( 2^n \) and, at every point of \( M \),
\[ \eta_1 \wedge \cdots \wedge \eta_s \wedge (d\eta_i)^n \neq 0. \]

It is known [1] that if \( M^{2n+s} \) has s-contact structure, then it has an \((f, E, \eta_i, g)\)-structure, which we call associated to the s-contact structure, such that \( F = d\eta_i \), where \( F \) is the fundamental 2-form. A normal \((f, E, \eta_i, g)\)-structure associated to an s-contact structure is called an S-structure. Notice that an S-structure is a K-structure.

**Theorem 4.1.** Let \( M^{2n+s} \) be a compact connected manifold with a regular S-structure \((f, E, \eta_i, g)\), \( i = 1, \ldots, s \). Then \( M^{2n+s} \) is the bundle space of a principal toroidal bundle over a Hodge manifold \( N^{2n} \).
PROOF. By Theorem 2.1 and its proof we have that $M^{2n+3}$ is the bundle space of a principal $T^s$-bundle over a Kaehler manifold $N^{2n}$, and that the group action is given by the one-parameter groups of transformations of the vector fields $E_1, \ldots, E_s$.

Now we claim that the form

$$\omega = \sum_{i=1}^{s} \eta_i \otimes e_i$$

is a connection form. This is, $\omega$ satisfies:

a) $R^*_i \omega = \omega$, for $i \in T^s$.

b) $\omega(X^*) = X$, where $X^*$ is the fundamental vector fields of $X$, with $X$ in the Lie algebra of $T^s$.

Part a) follows from the fact $L_{E_i} \eta^j = 0$, $i, j = 1, \ldots, s$, which is a consequence of the normality of the $S$-structure. For part b) it suffices to prove it for the vector $e_i$, $i = 1, \ldots, s$. But this follows immediately from $e_i^* = E_i$.

On the other hand, from the proof of Theorem 2.1, we also have that the fundamental form of the $f$-structure, $F$, and the fundamental form of the Kaehlerian structure, $\Omega^*$, are related by

$$F = \pi^* \Omega^*$$

where $\pi$ is bundle projection. But, in the particular case of an $S$-structure, we have $F = d\eta^i$, $i = 1, \ldots, s$. Therefore $d\eta^i = \pi^* \Omega^*$. Hence, by Theorem 3.1, $[\Omega^*]$ is $H(N, Z)_{\nu}$, which says that $N^{2n}$ is a Hodge manifold.

THEOREM 4.2. Let $M(N, T^s, \pi)$ be a principal toroidal bundle whose base space $N^{2n}$ has an almost Hermitian structure. Then $M$ has a regular $(f, E^*_\nu, \eta^i, g)$-structure, $i = 1, \ldots, s$.

PROOF. Fix a connection form $\omega = \sum_{i=1}^{s} \eta^i \otimes e_i$ on $M$ and let $E^*_\nu$ be the fundamental vector of $E^*_\nu$. Then we have

$$\eta^i(E^*_\nu) = \eta^i_{\nu}.$$

Let $(J, g')$ be the almost Hermitian structure of $N$. If $u \in M$, $\pi(u) = v$ and $\pi_\nu : T_v(N) \rightarrow T_u(M)$ is the lifting with respect to the fixed connection, define $f$ by

$$f(X) = (\pi_\nu \circ f \circ \pi_\nu)(X), \ X \in T_u(M).$$
Then we have \( f(E_i) = 0 \) and \( \eta^i \circ f = 0 \), \( i = 1, \ldots, s \). We also have
\[
f^2(X) = (\pi \circ j \circ \pi)^2(X) = -(\pi \circ \pi)(X) = -X + \sum_{i=1}^{s} \eta^i(X)E_i
\]
this is, \( f^2 = -I + \sum_{i=1}^{s} \eta^i \otimes E_i \). Thus we have an \((f, E_i, \eta^i)\)-structure, \( 1 \leq i \leq s \), on \( M \). Furthermore, the Riemannian metric \( g \) on \( M \) defined by
\[
g(X, Y) = g'(\pi X, \pi Y) + \sum_{i=1}^{s} \eta^i(X)\eta^i(Y)
\]
is associated to this \((f, E_i, \eta^i)\)-structure, since
\[
g(fX, fY) = g'(\pi fX, \pi fY) + \sum_{i=1}^{s} \eta^i(fX)\eta^i(fY)
= g'(f\pi X, f\pi Y) = g'(\pi X, \pi Y)
= g(X, Y) - \sum_{i=1}^{s} \eta^i(X)\eta^i(Y).
\]

It is clear from the definition of \( E_i \) that each one of these is regular. The regularity of the distribution determined by all the \( E_i \)'s (vertical distribution) follows from the Theorem XIV of [5], which says that if the leaf space of an integral distribution is a manifold and if the projection mapping takes the tangent space of any point onto the tangent space of its projection, then the distribution must be regular.

**Theorem 4.3.** *The framed \( f \)-structure defined in the previous theorem is normal if and only if the following two conditions hold:
1) \( f \) is a complex structure.
2) \( d\omega(fX, fY) = -d\omega(X, fY) \), for any \( X, Y \).*

**Proof.** Since 2) is equivalent to 3) \( d\omega(fX, fY) = d\omega(X, Y) \) the theorem will follow as soon as we prove the two equalities:

- a) \( \pi(S(X, Y)) = [J, f \pi X, \pi Y] : X, Y \) right invariant vector fields.
- b) \( \omega(SX, SY) = d\omega(X, Y) - d\omega(fX, fY) \), for any \( X, Y \).

a) If \( X, Y \) are right invariant vector fields on \( M \), so are \([X, Y], f(X) \) and \( f(Y) \). (\( f \) is right invariant). Besides, we have the relations:
\[
\pi[X, Y] = [\pi X, \pi Y] \quad \text{and} \quad \pi f = f \circ \pi.
\]
Therefore
\[
\pi(S(X, Y)) = \pi([f, f \pi X, \pi Y] + \Sigma d\eta^i(X, Y)E_i) = [J, f \pi X, \pi Y].
\]
b) Since \( f \) is horizontal we have \( d\omega(fX, fY) = -\omega([fX, fY]) \). Hence
\[ \omega(S(X, Y)) = \omega([fX, fY]) + d\omega(fX, fY) + d\omega(X, Y). \]

**Theorem 4.4.** Let \( N^{2n} \) be a Hodge manifold. Then for each \( s \geq 1 \) there exists a principal toroidal bundle \( M(N, T^s, \pi) \), whose bundle space \( M^{2n+s} \) has a regular S-structure.

**Proof.** Let \((J, g')\) be the Hodge structure on \( N \), and \( \Omega^* \) its fundamental 2-form. Since \([\Omega^*] \in H^2(N, \mathbb{Z})_p\), then
\[
([\Omega^*], \ldots, [\Omega^*]) \in \bigoplus_{i=1}^{s} H^2(N, \mathbb{Z})_p.
\]

By Theorem 3.1, there exists a toroidal bundle \( M=M(N, T^s, \pi) \) such that \( \mathcal{F}(M) = ([\Omega^*], \ldots, [\Omega^*]) \). We can find a connection form \( \omega = \sum_{i=1}^{s} \eta^i \otimes e_i \) whose curvature from \( d\omega \) satisfies
\[
d\omega = \sum_{i=1}^{s} d\eta^i \otimes e_i = \sum_{i=1}^{s} \pi^* \Omega^* \otimes e_i.
\]
The forms \( \eta^1, \ldots, \eta^s \) define a \( s \)-contact structure on \( M^{2n+s} \). In fact, since \( d\eta^i = \pi^* \Omega^* \), the rank of \( d\eta^i \) is \( 2n \).

On the other hand, if \( E_1, \ldots, E_s \) are the fundamental vector fields of \( e_1, \ldots, e_s \), we have \( \eta^i(E_j) = \delta^i_j \). Now, taking \( E_1, \ldots, E_s \) and \( X_1, \ldots, X_{2n} \) horizontal and linearly independent vectors, we get
\[
\eta^{1 \wedge \ldots \wedge s \wedge (d\eta^i)^n} (E_1, \ldots, E_s, X_1, \ldots, X_{2n})
= (d\eta^i)^n (X_1, \ldots, X_{2n}) = \Omega^* (\pi X, \ldots, \pi X_{2n}) \neq 0
\]
which proves that \( \eta^{1 \wedge \ldots \wedge s \wedge (d\eta^i)^n} \neq 0 \) at every point of \( M \).

If \((f, E_i \eta^i, g)\) is the framed \( f \)-structure on \( M \) constructed in the Theorem 4.2 using the Hodge structure \((J, g')\) on \( N \), we have
\[
F(X, Y) = g(X, fY) = g'(\pi X, \pi fY) = g'(\pi X, f\pi Y) = \Omega^*(\pi X, \pi Y) = d\eta^f(X, Y).
\]

Therefore this \((f, E_i \eta^i, g)\)-structure is associated to this \( s \)-contact structure defined by \( \eta^1, \ldots, \eta^s \). By Theorem 4.2 and its proof, \((f, E_i \eta^i, g)\) is regular. On the other hand, since \( f \) is a complex structure and \( d\omega(fX, fY) = d\omega(X, Y) \), \((f, E_i \eta^i, g)\) is normal, and therefore a regular S-structure on \( M \).

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REFERENCES


