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A Boundary Integral Approximation for the Stress Intensity Factors in Elastic Plate Bending

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彈性平板굽힘 問題의 應力擴大係數를 위한 境界積分數値解法

金 珍 佑

抄 錄

크랙彈性平板의 굽힘問題가 境界積分方程式으로 構成되었다. 自然變數인 變位, 垂直기울기, 굽힘 모우멘트, 等價剪斷力과 크랙끝에서 應力의 成長率로 定義되는 應力擴大係數들이 主變數로 포함된다. 이 積分方程式들은 可逆에너지 積分理論(Green-Rayleigh)을 기초로 크랙應力分布特性에 맞게 발전되었으며 해당되는 核函數들이 誘導되었다. 等分布 모우멘트를 받는 中央크랙이 있는 정 4 각형 모형에 대한 應力擴大係數가 計算되어 기존의 有限要素法의 解와 비교되었다.

1. Introduction

A fairly general boundary integral equation (BIE) formulation for elastic plate flexure has been given earlier by Stern(1) in terms of a pair of coupled singular integral equations involving the natural variables of deflection, normal slope, bending moment, and equivalent shear on the plate boundary. While this formulation allows for discontinuities in the boundary variables as might naturally occur at a corner of the plate boundary. or where the boundary support conditions undergo a sudden change in type as from clamped to free, the class of admissible problems is still required to have bounded moment and shear resultants which produce bounded stresses everywhere in the plate. However, there are important classes of problems for which the moment and shear resultants as calculated within the framework of linear theory are not bounded, for example at sharp notches or cracks as shown by Williams(2). At such singular points a knowledge of the so-called stress intensity factors governing the growth rate of the stresses has proven useful in linear fracture mechanics.

In this chapter we indicate how to adapt the BIE formulation so that the stress intensity factors also become natural variables to be determined by the solution of coupled singular integral equations. While the basic method follows closely in spirit the ideas outlined for plane elastostatic calculations by Barone and Robinson(3) and by Stern(4), the complexity of the problem is magnified by the higher-

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order equations of plate theory, and some new difficulties not present in the plane elastostatic case arise.

2. Boundary Integral Representation

We first briefly summarize the development of the BIE representation in Stern(1). The plate is modeled by a bounded region Ω with total boundary $\partial\Omega$ containing a finite number of corner points $l_1 \cdots l_k$ as indicated in Fig. 1 where other relevant notation is shown. The plate deflection w is governed by the differential equation

$$\nabla^4 w = q/D \text{ in } \Omega \tag{1}$$

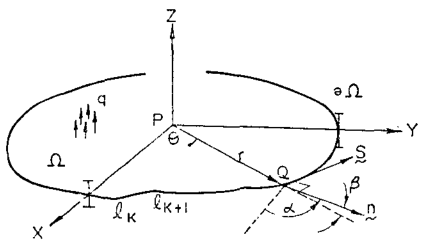

Fig. 1 Coordinates and notation.

where q is the transverse load on the plate, and suitable boundary conditions, left unspecified at the moment, are imposed on $\partial\Omega$. Also $D = Eh^3/12(1-\nu^2)$ denotes the flexural rigidity of the plate and ∇^4 is the iterated Laplacian

For u any other sufficiently smooth function on Ω, interpretable as a possible deflection for suitable loading and support conditions, we have a natural reciprocal work identity in the form

$$\int_{\partial\Omega} \{V_n(u)w - M_n(u)\frac{dw}{dn} + \frac{du}{dn}M_n(w) - uV_n(W)\}\, ds + \sum_{k=1}^{K}\Big[[\![M_t(u)]\!]w - [\![M_t(w)]\!]u\Big]_{l_k} = D\int_{\Omega}(u\nabla^4 w - w\nabla^4 u)da \tag{2}$$

The notation and development was detailed in Stern(1); briefly $V_n(\cdot)$, $M_n(\cdot)$, $\frac{d\cdot}{dn}$ are the equivalent shear, bending moment and normal slope on the boundary $\partial\Omega$, and $[\![M_t(\cdot)]\!]_{l_k}$ is the discontinuity jump in twisting moment (interpertable as a concentrated force reaction) at the corner l_k.

Now let $P \varepsilon \Omega$ be an interior point of the plate (the origin of coordinates in Fig. 1) and introduce the special "singluar solution"

$$w(P) = \frac{1}{8\pi D} r^2 \ln r \tag{3}$$

with N, V, M and T the corresponding normal slope, equivalent shear, bending moment and twisting moment on $\partial\Omega$. Deleting a small circular region centered at P, and applying the identity $Eq.$ (2) to Ω so modified, produces two contributions to the boundary integral. The first integral is still over the entire boundary $\partial\Omega$ just as in Eq. (2). The second however is over the small circle surrounding P. With the particular choice of the function W defined in Eq. (3) for the auxiliary functionn u, this last integral may be evaluated in the limit as the circle surrounding P shrinks to P.

This produces, from Eq. (2), the representation

$$w|_P - \int_{\partial\Omega}\Big[Vw - M\frac{dw}{dn} + NM_n(w) - WV_n(w)\Big]ds - \sum_{k=1}^{K}\Big[[\![T]\!]w - [\![M_t(w)]\!]W\Big]_{l_k} + \int_{\Omega} qWda \tag{4}$$

where we have replaced $D\nabla^4 w$ with the distributed load intensity q. Since only the functions derived from W depend on the point P, Eq. (4) may be differentiated (in the interior of Ω) to whatever extent is permitted by the regularity of the load q. Thus the plate deflection and quantities derivable from it are determined by a knowledge of the deflection, normal slope, bending moment, and equivalent shear on the plate boundary, and the discontinuity jumps in the twisting moment(concentrated support forces) which might occur at corners. These quantities in turn may be determined using the following considerations.

The same type of argument used in obtaining Eq. (4) is repeated for the point P on the boundary $\partial\Omega$. Now only a semi-circular region is deleted from Ω and two new corners are introduced into the boundary of the modified region as indicated in Fig. 2. In addition to the special singular function W defined in Eq. (2), we define another

$$W_\gamma=\frac{1}{2\pi D}r\ln r\cos(\theta+\gamma) \qquad (5)$$

with γ the angle from the outer normal at P to the line $\theta=0$, and N_γ, V_γ, M_γ, T_γ the corresponding normal slope, equivalent shear, bending moment and twisting moment. This leads to the pair of coupled boundary integral representations

$$\frac{1}{2}w\Big|_P+\oint_{\partial\Omega}\Big[Vw-M\frac{dw}{dn}+NM_n(w)-WV_n(w)\Big]ds+\sum_{k=1}^{K}\Big[[\![T]\!]w-[\![M_t(w)]\!]W\Big]_{l_k}=\int_\Omega qWda \qquad (6)$$

$$\frac{dw}{dn}\Big|_P+\oint_{\partial\Omega}\Big[V(w-w\Big|_P)-M_\gamma\frac{dw}{dn}+N_\gamma M_n(w)-W_\gamma V_n(w)\Big]ds+\sum_{k=1}^{K}\Big[[\![T_\gamma]\!](w-w\Big|_P)-[\![M_t(w)]\!]W_\gamma\Big]_{l_k}=\int_\Omega qW_\gamma da \qquad (7)$$

where $\oint$ denotes a Cauchy principal value integral. If the point P should lie at a corner of the plate boundary additional considerations are required; these are developed in detail in Stern(1). Finally, two boundary conditions involving the boundary variables are available from the particualr nature of the supports or lack of them; for example, on a clamped portion of the boundary the deflection w and normal slope $\frac{dw}{dn}$ are required to vanish, whereas if the boundary is free of support then the bending moment $M(w)$ and equivalent shear $V_n(w)$ are zero. The boundary conditions, together with Eqs. (6) and (7), are sufficient to determine the boundary variables everywhere on $\partial\Omega$.

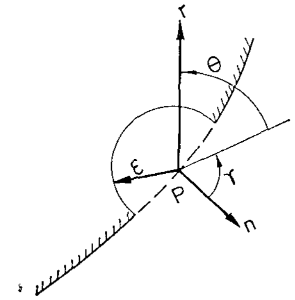

Fig. 2 Origin on the boundary.

3. Singularity Equations

The equations obtained in the preceeding section are based on the presumption that the moment and shear resultants remain bounded near the origin point P. We now consider how to obtain appropriate integral equations for other cases of interest at boundary points where the stresses bcome unbounded. For brevity we outline the major ideas only for the special case of a through crack with free edges.

Following the ideas of Williams(2), we infer the asymptotic singular behavior of the plate deflection at the base of a crack from the nontrivial solutions of the homogeneous boundary value problem corresponding to an unloaded plate containing a straight semi-infinite crack with free edges:

$$\nabla^4 w(r,\ \theta)=0 \text{ for } r>0,\ 0\le\theta\le2\pi$$

$$M_n(w)=V_n(w)=0 \text{ for } \theta=0\Big(\beta=\frac{3\pi}{2}\Big) \qquad (8)$$

$$M_n(w)=V_n(w)=0 \text{ for } \theta=2\pi\Big(\beta=\frac{\pi}{2}\Big)$$

A separation of variables solution results in an eigenvalue problem; with

$$w(r,\theta;\lambda)=r^{\lambda+1}[b_1^{(\lambda)}\sin(\lambda+1)\theta+b^{2(\lambda)}\cos(\lambda+1)\theta+b^{3(\lambda)}\sin(\lambda-1)\theta+b_4^{(\lambda)}\cos(\lambda-1)\theta]$$

(9)

the eigenvalues are given by

$$\sin 2\pi\lambda=0 \tag{10}$$

We reject negative values of λ as leading to unbounded strain energy, while for $\lambda=0$, $\lambda=1$ or $\lambda\geq 2$ the stresses remain bounded. The acceptable eigensolutions with singular stresses thus correspond to the eigenvalues $\lambda=1/2$ and $\lambda=3/2$ which leads to the general singular solution

$$\begin{aligned} w^s(r,\theta)=r^{3/2}&\left\{b_1\left[\sin\frac{3\theta}{2}+\frac{3(1-\nu)}{7+\nu}\sin\frac{\theta}{2}\right]\right.\\ &\left.+b_2\left[\cos\frac{3\theta}{2}+\frac{3(1-\nu)}{5+3\nu}\cos\frac{\theta}{2}\right]\right\}\\ +r^{5/2}&\left\{c_1\left[\sin\frac{5\theta}{2}-\frac{5(1-\nu)}{9-\nu}\sin\frac{\theta}{2}\right]\right.\\ &\left.+c_2\left[\cos\frac{5\theta}{2}+\frac{5(1-\nu)}{3+5\nu}\cos\frac{\theta}{2}\right]\right\} \end{aligned} \tag{11}$$

Equation(11) defines the asymptotic singular behavior of the plate deflection near the base of a crack in terms of the four parameters b_1, b_2, c_1, c_2. (This result is of course well known, for example Williams(5)

The idea now is to substitute for u in the reciprocal work identity Eq. (2) particular biharmonic functions with the right order of singular behavior at the base of the crack, so that when the argument leading to the representations Eqs.(6) and(7) is repeated for P at the crack tip we obtain representations for b_1, b, c_1, c_2. Furthermore, if these functions are also solutions of the boundary value problem Eq. (8) then no contribution to the integrals will result from the crack flanks.

Without attempting to furnish details we merely list main results. The particular solution of Eq. (8) required(called the complementary solution) is of the form

$$\begin{aligned} u(r,\theta)=r^{1/2}&\left\{B_1\left[\sin\frac{\theta}{2}+\frac{1-\nu}{5+3\nu}\sin\frac{3\theta}{2}\right]\right.\\ &\left.+B_2\left[\cos\frac{\theta}{2}+\frac{1-\nu}{7+\nu}\cos\frac{3\theta}{2}\right]\right\}\\ +r^{-1/2}&\left\{C_1\left[\sin\frac{\theta}{2}+\frac{1-\nu}{3+5\nu}\sin\frac{5\theta}{2}\right]\right.\\ &\left.+C_2\left[-\cos\frac{\theta}{2}+\frac{1-\nu}{9-\nu}\cos\frac{5\theta}{2}\right]\right\} \end{aligned} \tag{12}$$

where B_1, B_2, C_1, C_2 are arbitrary constants. By deleting a small circular region surrounding the crack tip at P as indicated in Fig. 3 we add to Eq. (2) a contour integral(on C_ε) and two corner jump terms (at l^+ and l^-) which are evaluated in the limit as $\varepsilon\to 0$ to produce, after routine but tedious calculation,

$$\begin{aligned} J_{tip}=\lim_{\varepsilon\to 0}&\left\{\int_{C_\varepsilon}\left[V_n(u)w-M_n(u)\frac{dw}{dn}+\frac{du}{dn}M_n(w)-uV_n(w)\right]ds\right.\\ &+\left[[\![M_t(u)]\!]w-[\![M_t(w)]\!]u\right]_{l^+}\\ &\left.+\left[[\![M_t(u)]\!]w-[\![M_t(w)]\!]u\right]_{l^-}\right\}\\ &=\frac{24(1-\nu)(3+\nu)\pi D}{(7+\nu)(5+3\nu)}[B_1b_1+B_2b_2]\\ &-\frac{120(1-\nu)(3+\nu)\pi D}{(9-\nu)(5+3\nu)}[C_1c_1+C_2c_2] \end{aligned} \tag{13}$$

By suitable choice of the arbitrary constants B_1, B_2, C_1, C_2 in the complementary solution Eq. (12) we can obtain representations analogous to Eqs. (6) and (7) for the parameters b_1, b_2, c_1, c_2. Alternatively, more physically meaningful parameters might be introduced in place of these; for example a symmetric moment intensity factor $K_1^{(1)}$ which governs the rate of growth of the bending moment:

$$K_1^{(1)}=\lim_{r\to 0} r^{1/2}M_n(w)\Big|_{\substack{\theta=\pi\\(\beta=\pi/2)}}=-\frac{3D(1-\nu)(3+\nu)}{7+\nu}b_1 \tag{14}$$

An antisymmetric moment intensity factor may also be defined and satisfies

$$K_2^{(1)}=\lim_{r\to 0} r^{1/2}M_t(w)\Big|_{\substack{\theta=\pi\\(\beta=\pi/2)}}=-\frac{3D(1-\nu)(1+\nu)}{5+3\nu}b_2 \tag{15}$$

Then, for example, if we take

$$B_1=-\frac{5+3\nu}{8\pi},\quad B_2=C_1=C_2=0 \tag{16}$$

in Eq.(12) (denoting this special function u_1)

we find Eq. (13) reduces to $J_{tip}=K_1^{(1)}$ and the representation becomes

$$K_1^{(1)}+\oint_{\partial\Omega}\Big[V_n(u_1)w-M(u_1)\frac{dw}{dn}+\frac{du_1}{dn}M_n(w)-u_1V_n(w)\Big]ds+\sum_{k=1}^{K}\Big[[\![M_t(u_1)]\!]w-[\![M_t(w]\!]u_1\Big]_{l_k}=D\int_{\Omega}u_1 q\,da \tag{17}$$

with similar equations for $K_2^{(1)}$, c_1 and c_2

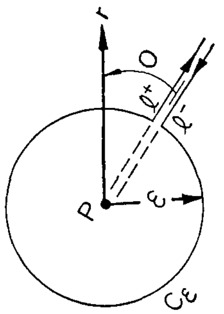

Fig. 3 Integratinon near crack tip.

4. Numerical Implementation

The incorporation of the additional singularity intensity variables in the numerical BIE scheme described in Stern (1) involves several new considerations, Briefly the original scheme consists of discretizing the boundary by the introduction of a finite number of nodal points, care being taken to place a nodal point at corners and other exceptional points. The primary variables are then interpolated between nodal points which reduces the problem to the determination of nodal values of each of the four primary variables(deflection, normal slope, bending moment and equivalent shear) at the "regular nodes", with corners requiring some added considerations as detailed in Stern (1). We have generally at each node two boundary conditions to be imposed as well as the pair of integral equations (6) and (7) which are easily discretized in terms of the nodal variables by any convenient quadrature rule applied to the intervals between adjacent nodes. The solution of this system of linear equations (including additional corner variables and equations) then completes the process.

Now we have introduced singularity intensity factors as additional unknown variables, and with each an additional integral equation such as Eq. (17) is also furnished so that the system of equations is still determinate. However, some as yet untreated questions arise in the limiting process leading to the integral equations for origin at the singularity.

The evaluation of various limits in obtaining Eqs. (6) and (7) was accomplished with the presumption that the deflection function behaved smoothly (bounded third derivatives) near the boundary of the plate. This of course is not the case near a singular point of the type under consideration. It is not much more difficult to verify these limits even in the singular case if one notes the form of the asymptotic behavior of the plate deflection near the singularity, for example Eq. (11) at the base of a through crack.

A similar question arises in deriving representations such as Eq. (17) since the deflection and rotation of the plate at the crack tip were (tacitly) assumed to be zero. Again it can be verified by direct calculation(as well as by an energy argument) that the singularity intensity equations are unaltered by any superposd rigid body displacement of the plate.

Finally, the boundary segments adjacent to a singular node should receive special treatment since the eigensolutions furnish additional information concerning the behavior of the primary variables. For example consider the segment from the crack tip to the adjacent node on the crack edge a distance d away. Normally we would interpolate the displacement on this interval linearly writing

$$w(r)=w_0+(w_1-w_0)r/d \tag{18}$$

where w_0 is the deflection at the crack tip node and w_1 is at the adjacent node. However, from Eq. (11), with whatever rigid body displacement of the plate is needed at the crack tip, the deflection along this segment must be of the form

$$w(r)=w_0+w_0'r+pr^{3/2}+qr^{5/2}+\text{remainder} \tag{19}$$

where w_0' is the slope along the crack edge at the tip, with a corresponding representation similar to Eq. (7), and p, q are particular linear combinations of the intensity parameters K_1, K_2, c_1, c_2. The remainder term may be approximated to the order of r^2 so that the deflection is continuous at the adjacent node yielding the interpolation

$$w(r)=w_0+w_0'r+pr^{3/2}+qr^{5/2} + \{w_1-w_0+w_0'd+pd^{3/2}+qd^{5/2}\}\ (r/d)^2 \tag{20}$$

Similar results are obtained to interpolate the other variables.

As an example of the entire procedure consider the symmetric bending of a centrally cracked square plate as illustrated in Fig. 4. Symmetric considerations permit us to analyze only one quarter of the plate(the second quadrant, for example which is isolated in Fig.5. Boundary conditions are also shown in Fig.5.

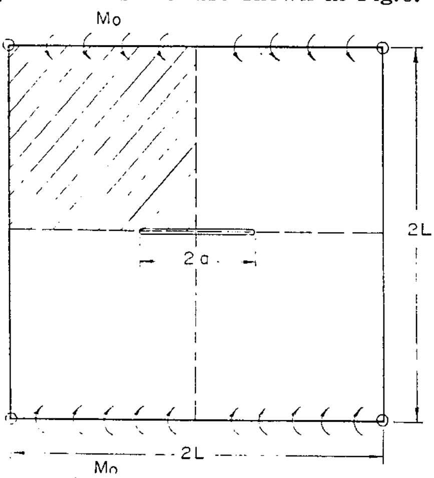

Fig. 4 Symmetric bending of a centrally cracked square plate.

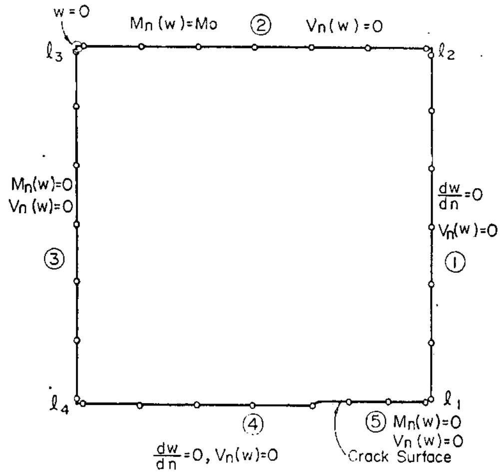

Fig. 5 Nodal distribution and boundary conditions for one quarter of the plate.

Results for the symmetric moment intensity factor $K_1^{(1)}$ are shown in Table 1. for five ratios of a/L ranging from. 5 to. 1. An eight node mesh on every side was used in all five cases. The results are nondimensionalized using the solution for an infinite plate (a/L→0) obtained by Sih, et al (6). These nondimensional intensity factors for a square plate are compared with those of Wilson's finite element solution(7) for an infinite strip of finite width with symmetric uniform bending. As indicated in Table 1. there is reasonable agreement but the BIE solutions exhibit rather poor behavior over the range of crack size. This may be due to inherent poor conditioning of the BIE equations and the delicate nature of the CPV evaluations.

While the numerical results appear to be generally correct, they are far from satisfactory. In the numerical solution there were indications that the system of equations was poorly conditioned. It was also found that the results were sensitive to the form of interpolation used on the elements adjacent to the crack tip and in the CPV evaluations. Further study is needed to identfy and correct, or at least to minimize the effects of the major error mechanisms. Numerical experiments with different geometries, mesh patterns and interpolation forms could be useful.

Additional areas also merit investigation. The reatment of curved boundaries should be investigated to determine whether significant errors are introduced by crude approximation (say piecewise linear) of its shape. In principle, the shape of the boundary may be treated as accurately as desired, but with a significant increase in computation cost. Another area of investigation which might prove fruitful is the possible gain in accuracy for a given mesh using a smoother than piecewise linear representation of the boundary variables, for example cubic splines. There are no theoretcal analyses presently available to suggest how the accuracy of the solution depends on the smoothness of the boundary approximations, so numerical experiments will probably be the major investigative tool here as well.

Table 1 Nondimensional symmetric moment intensity factors.

a/L	.1	.2	.3	.4	.5
$K_1^{(1)}/K^\infty$ (BIE)	1.0130	1.0480	1.0578	1.1011	1.2117
$K_1^{(1)}/K^\infty$ (FEM)	1.006	1.024	1.058	1.105	1.181

Acknowledgements

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