Partially Balanced Resolution Ⅳ* Designs in a 2^m-Factorial

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ABSTRACT

Srivastava and Anderson (1970) illustrate a method of obtaining Balanced (but not orthogonal) Resolution Ⅳ* designs starting with a BIB design. The incidence matrix of a BIB design with parameters (v, b, r, k, and λ) is utilized to obtain Balanced Resolution Ⅳ* designs with m factors and $n=2b$ runs, where $m \leq v$. In this paper, the same idea is extended to the case of PBIB designs to obtain Partially Balanced Resolution Ⅳ* designs. In the designs obtained here the variances are balanced and the covariances are partially balanced with respect to the main effects. A proof of this property of Partially Balanced Resolution Ⅳ* designs is given. The efficiency of Partially Balanced Resolution Ⅳ* designs is also considered and examples of Partially Balanced Resolution Ⅳ* designs are included.

1. Introduction

Consider 2^m-fractional factorial designs of Resolution Ⅳ, where the three and highter order interactions are assumed negligible and the main effects are parameters of interest. The mean and the two-factor interactions are not assumed to be negligible and they are not of interest with regard to estimation. A subclass of the above is known as Resolution Ⅳ* designs if the main effects are estimable orthogonal to the general mean and two-factor interactions.

Using Box-Wilson fold-over principle, we can generate a Resolution Ⅳ* design from a Resolution Ⅲ design. It is well known that Resolution Ⅲ designs for 4α-1 factors in 4α runs can be obtained using Hadamard matrices. To each run then we adjoin a factor at level 1 and folding over the plan thus obtained we get a Resolution Ⅳ* design for 4α

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$(=m)$ factors in $8\alpha(=n)$ runs. These designs are orthogonal and balanced. However, orthogonal designs, while a subclass of the class of balanced designs, are not available for every value of n.

Srivastava and Anderson (1970) illustrated a method of obtaining Balanced (but not orthogonal) Resolution Ⅳ* designs with m factors and $n=2b$ runs starting with a BIB design with parameters $(v, b, r, k$, and $\lambda)$, where $m \leq v$.

The same idea could be extended to the case of PBIB designs to obtain Partially Balanced Resolution Ⅳ* designs. In this paper, we deal with Partially Balanced Resolution Ⅳ* designs. In Section 2, we illustrate a method of a Partially Balanced Resolution Ⅳ* design using a PBIB design and the property of this design is investigated. In Section 3, we consider efficiency of Partially Balanced Resolution Ⅳ* designs. Examples of Partially Balanced Resolution Ⅳ* designs with m factors in $2m$ runs are presented in Section 4. A result in Section 2 is proved in Appendix.

2. Partially Balanced Resolution Ⅳ* design in a 2^m-factorial

We can illustrate a method of obtaining a Partially Balanced Resolution Ⅳ* design m factors in $n=2b$ runs using a PBIB design with parameters $(v=m, b, r, k;\ \lambda_1, \lambda_2, \cdots, \lambda_t)$. Write down the incidence matrix N(mxb), and we get a design T given by

(2.1) $$T=[N : \bar{N}],$$

where $\bar{N}$ denotes the matrix by interchanging the symbols 0 and 1 in N. This is the well known Box-Wilson fold-over principle, so the design T is a Resolution Ⅳ* design in a 2^m-factorial.

Webb(1968) showed that for any Resolution Ⅳ design, we have $n \geq 2m$; Margolin(1969) independently proved Webb's result, and also showed that if $n=2m$, and if every factor occurs at level 0 as often as at level 1, then a Resolution Ⅳ desgin is a necessarily a fold-over design. Hence minimal Resolution Ⅳ* designs are fold-over types.

The coefficient matrix for the fold-over design (2.1) will be of the form

(2.2) $$X=\begin{pmatrix} \mathbf{1}_b & X_1 & X_2 \\ \mathbf{1}_b & -X_1 & X_2 \end{pmatrix}$$

where $\mathbf{1}_b$ is the bxl vector of 1's, $[X_1' : -X_1']'$ and $[X_2' : X_2']'$ are the coefficient matrices corresponding to the main effects and the two-fabtor interactions, respectively.

From (2.2)

(2.3) $$X'X=\begin{pmatrix} 2b & 0 & 21'X_2 \\ 0 & 2X_1'X_1 & 0 \\ 2X_2'1 & 0 & 2X_2'X_2 \end{pmatrix}$$

This means that the main effects are estimable orthogonal to the general mean and two-factor interactions, if $X_1'X_1$ is nonsingular.

We know that, from the incidence matrix N,

(2.4) $$X_1=2N'-J_{bm}'$$

where J_{bm} is the bxm matrix each of whose element is unity, so

(2.5) $$2X_1'X_1=8NN'+(2b-8r)J_{mm}.$$

Suppose that the matrix $X_1'X_1$ is nonsignular, then the variance-covariance matrix of the estimators of the main effects under $T=[N:\bar{N}]$ is

(2.6) $$V_T=\frac{1}{2}(X_1'X_1)^{-1}\sigma^2=\frac{1}{2}(4NN'+(b-4r)J_{mm})^{-1}\sigma^2,$$

Now, let B_i, $i=1,2,\cdots,t$, be the association matrices of the PBIB design in (2, 1), then it is well known that

(2.7) $$NN'=rB_0+\lambda_1B_1+\cdots+\lambda_tB_t,$$

where $B_0=I_m$ the $m\times m$ identity matrix. So the matrix $X_1'X_1$ in (2.6) can be expressed as follows: Since $B_0+B_1+\cdots B_t=J_{mm}$,

(2.8) $$X_1'X_1=c_0B_0+c_1B_1+\cdots+c_tB_t.$$

We will prove the following result in Appendix if the matrix $X_1'X_1$ is nonsingular

(2.9) $$(X_1'X_1)^{-1}=q_0B_0+q_1B_1+\cdots+q_tB_t$$

for real numbers q_i's.

This means that the variance-covariance matrix of the estimators of main effects under the design $T=[N:\bar{N}]$ is

(2.10) $$V_T=\frac{1}{2}\left(\sum_{i=0}^{t} q_iB_i\right)\sigma^2.$$

For a Partially Balanced Resolution Ⅳ* design constructed from a PBIB design with above parameters, therefore, the variance of the estimate of every main effect is the same, and the covariances between two of such estimators have t-associate partially balanced scheme in the following sense:

Let cov(i,j) be a covariance between i th and j th main effects corresponding to the i th and j th treatments respectively which are the k th associates in the original PBIB design, then the value of cov (i,j) is $\frac{1}{2}q_k\sigma^2$ and this uniquely determined only by the associates. In this sense the above design (2.1) can be called Partially Balanced Resolution

Ⅳ* design with parameters $(m=v,\ n=2b;\mu_0,\mu_1,\cdots,\mu_t)$, where $\mu_i=\frac{1}{2}q_i$, $i=0,1,\cdots,t$.

3. Efficiency of Partially Balanced Resolution Ⅳ* degign

In determining the efficiency of Balanced Resolution Ⅳ* design, Srivastava and Anderson (1970) presented the following results.

Let B_n denotes the class of all balanced nonsingular fractional designs of Resolution Ⅳ* with m factors and n runs. Let $T\in B_n$, then the lower bound of $\mathrm{tr}(V_T)$ is m/n, if $n=4\alpha$, and $(m-1)/(n-2)+1/(n-2+2m)$, if $n=4\alpha+2$, where α is an integer. Also they proved that(a) T is trace optimal in B_n, if T is determinant optimal and conversely, (b) T is maximum root optimal in B_n, if T is trace optimal and conversely.

Finally, they defined the following efficiency measure for Balanced Resolution Ⅳ* designs:

(3. 1) $$\text{absolute (trace) efficiency}=\tau(m,n)/tr(V_T),$$

where

(3. 2) $$\tau(m,n)=\begin{cases} m/n, & \text{if } n=4\alpha, \\ (m-1)/(n-2)+1/(n-2+2m), & \text{if } n=4\alpha+2 \end{cases}$$

where α is an integer.

We could use the above efficiency measure (3. 1) for Partially Balanced designs of Resolution Ⅳ*, where, in this case, $\tau(m,n)$ is lower bound defined above given m and n and V_T is the variance-covariance matrix of our concerning Partially Balanced design of Resolution Ⅳ*. The ratio $\tau(m,n)/tr(V_T)$ therefore measures the efficiency of our design relative to a(possibly nonexistent) optimum Balanced Resolutio n Ⅳ* design given m and n.

4. Examples

We will present examples for 9 factors with 18 runs and 10 factors with 20 runs, because these designs are not appeared in Srivastava and Anderson (1970).

Example 4. 1 Consider a group divisible PBIB design with parameters $(v=9,\ b=9,\ r=7,\ k=7;\ \lambda_1=6,\ \lambda_2=5)$, whose blocks are (1, 2, 3, 4, 5, 6, 7), (2, 4, 5, 6, 7, 8, 9), (1, 2, 3, 5, 7, 8, 9), (3, 4, 5, 6, 7, 8, 9), (1, 2, 3, 4, 5, 6, 9), (1, 2, 3, 4, 5, 6, 8), (1, 2, 3, 6, 7, 8, 9), (1, 2, 3, 4, 7, 8, 9), and (1, 4, 5, 6, 7, 8, 9).

Then the incidence matrix N of the design is given by

$$N=\begin{pmatrix} 1&0&1&0&1&1&1&1&1\\ 1&1&1&0&1&1&1&1&0\\ 1&0&1&1&1&1&1&1&0\\ 1&1&0&1&1&1&0&1&1\\ 1&1&1&1&1&1&0&0&1\\ 1&1&0&1&1&1&1&0&1\\ 1&1&1&1&0&0&1&1&1\\ 0&1&1&1&0&1&1&1&1\\ 0&1&1&1&1&0&1&1&1 \end{pmatrix}$$

so

$$NN'=\begin{pmatrix} 7&6&6&5&5&5&5&5&5\\ 6&7&6&5&5&5&5&5&5\\ 6&6&7&5&5&5&5&5&5\\ 5&5&5&7&6&6&5&5&5\\ 5&5&5&6&7&6&5&5&5\\ 5&5&5&6&6&7&5&5&5\\ 5&5&5&5&5&5&7&6&6\\ 5&5&5&5&5&5&6&7&6\\ 5&5&5&5&5&5&6&6&7 \end{pmatrix}=I_3\otimes I_3+I_3\otimes J_3+5J_3\otimes J_3,$$

where symbol $\otimes$ stands for the Kronecker product.

Using matrix N, we obtain the following fold-over design T of 9 factors in 18 runs:

$$T=[N\,\vdots\,\bar{N}]$$

$$=\begin{pmatrix} 1&0&1&0&1&1&1&1&1&0&1&0&1&0&0&0&0&0\\ 1&1&1&0&1&1&1&1&0&0&0&0&1&0&0&0&0&1\\ 1&0&1&1&1&1&1&1&0&0&1&0&0&0&0&0&0&1\\ 1&1&0&1&1&1&0&1&1&0&0&1&0&0&0&1&0&0\\ 1&1&1&1&1&1&0&0&1&0&0&0&0&0&0&1&1&0\\ 1&1&0&1&1&1&1&0&1&0&0&1&0&0&0&0&1&0\\ 1&1&1&1&0&0&1&1&1&0&0&0&0&1&1&0&0&0\\ 0&1&1&1&0&1&1&1&1&1&0&0&0&1&0&0&0&0\\ 0&1&1&1&1&0&1&1&1&1&0&0&0&0&1&0&0&0 \end{pmatrix}$$

Each of the 18 columns of the above array corresponds to one run or treatment combination, and each of the nine rows to one of the nine factors. Thus, in the first run, the factors 1, 2, 3, 4, 5, 6, and 7 are at level 1, and the rest at level 0. This design T is a Resolution Ⅳ* derign with 9 factors and 18 runs in a 2^9-factorial since the design is a fold-over design.

In the design, from (2.5)

$$X_1'X_1=4NN'-19J_9$$
$$=4I_3\otimes I_3+4I_3\otimes J_3+J_3\otimes J_3,$$

so the matrix $X_1'X_1$ has Property A defined by Kurkjian and Zelen (1963).

or

$$X_1'X_1=9B_0+5B_1+B_2,$$

where $B_0=I_9$, $B_1=I_3\otimes(J_3-I_3)$, and $B_2=(J_3-I_3)\otimes J_3$.

We obtain

$$(X_1'X_1)^{-1}=\frac{1}{400}(100I_9-25I_3\otimes J_3-J_9)$$
$$=\frac{1}{400}(74B_0-26B_1-B_2)$$

Therefore, the design T is a Partially Balanced Resolution Ⅳ* design with parameters ($m=9$, $n=18$; $\mu_0=37/400$, $\mu_1=-13/400$, $\mu_2=-1/800$) in a 2^9-factorial.

In the design

$$tr(V_T)=74\times 9/800=333/400$$

and therefore, since in this case $\tau(m,n)=9/17$,

absolute (trace) efficiency$=9\times 400/17\times 333=0.64$.

Note: There exists BIB design with parameters ($v=9$, $b=9$, $r=8$, $k=8$, $\lambda=7$), whose blocks are (1, 2, 3, 4, 5, 6, 7, 8, 9), (1, 2, 4, 5, 6, 7, 8, 9), (1, 2, 3, 5, 6, 7, 8, 9), (1, 2, 3, 4, 6, 7, 8, 9), (1, 2, 3, 4, 5, 7, 8, 9), (1, 2, 3, 4, 5, 6, 8, 9), (1, 2, 3, 4, 5, 6, 7, 9), (1, 2, 3, 4, 5, 6, 7, 8), and (2, 3, 4, 5, 6, 7, 8, 9). However, the efficiency of the Balanced Resolution Ⅳ* design constructed from the BIB design is 0.52 which is less than the efficiency of the above Partially Balanced Resolution Ⅳ* design.

Example 4.2. Consider a triangular association scheme PBIB design with parameters ($v=10$, $b=10$, $r=4$, $k=4$; $\lambda_1=1$, $\lambda_2=2$), whose blocks are (2, 6, 7, 10), (1, 2, 5, 10), (2, 3, 7, 8), (2, 4, 6, 9), (1, 8, 9, 10), (3, 4, 5, 10), (1, 4, 7, 8), (3, 5, 7, 9), (1, 3, 6, 9), and (4, 5, 6, 8). In this design, $n_1=6$, $n_2=3$; $p^1_{11}=3$, $p^1_{12}=p^1_{21}=2$, $p^1_{22}=1$, $p^2_{11}=4$, $p^2_{12}=p^2_{21}=2$, $p^2_{22}=0$.

The association matrices are as follows

$$B_0=I_{10},\quad B_2=J_{10}-B_1-B_0,$$

where

$$B_1=\begin{pmatrix} 0&1&1&1&1&1&1&0&0&0\\ 1&0&1&1&1&0&0&1&1&0\\ 1&1&0&1&0&0&1&0&1&0\\ 1&1&1&0&0&0&1&0&1&1\\ 1&1&0&0&0&1&1&1&1&0\\ 0&1&0&1&0&0&1&1&0&1\\ 1&0&0&1&1&1&0&0&1&1\\ 0&1&1&0&1&1&0&0&1&1\\ 0&1&0&1&1&0&1&1&0&1\\ 0&0&1&1&0&1&1&1&1&0 \end{pmatrix}$$

In this case, we obtain the following Partially Balanced Resolution Ⅳ* design of 10 factors in 20 runs in a 2^{10}-factorial:

$$T=\begin{pmatrix} 0&1&0&0&1&0&1&0&1&0&1&0&1&1&0&1&0&1&0&1\\ 1&1&1&1&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1\\ 0&0&1&0&0&1&0&1&1&0&1&1&0&1&1&0&1&0&0&1\\ 1&0&0&1&0&1&1&0&0&1&0&1&1&0&1&0&0&1&1&0\\ 0&1&0&0&0&1&0&1&0&1&1&0&1&1&1&0&1&0&1&0\\ 1&0&0&1&0&0&0&0&1&1&0&1&1&0&1&1&1&1&0&0\\ 0&0&1&0&0&0&1&1&0&0&1&1&0&1&1&1&0&0&1&1\\ 0&0&1&0&1&0&1&0&0&1&1&1&0&1&0&1&0&1&1&0\\ 0&0&0&1&1&0&0&1&1&0&1&1&1&0&0&1&1&0&0&1\\ 1&1&0&0&1&1&0&0&0&0&0&0&1&1&0&0&1&1&1&1 \end{pmatrix}$$

In this design

$$X_1'X_2=4NN'-6J_{10}$$
$$=10B_0-2B_1+2B_2.$$

Therefore, from(a. 14) in Appendix

$$(10\mathscr{P}_0'-2\mathscr{P}_1'+2\mathscr{P}_2')\begin{pmatrix} q_0\\ q_1\\ q_2 \end{pmatrix}=\begin{pmatrix} 1\\ 0\\ 0 \end{pmatrix}$$

where

$$\mathscr{P}_0'=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad \mathscr{P}_1'=\begin{pmatrix}0&6&0\\1&3&2\\0&4&2\end{pmatrix},\quad \mathscr{P}_2'=\begin{pmatrix}0&0&3\\0&2&1\\1&2&0\end{pmatrix}.$$

We obtain the following solution:

$$q_0=5/32,\quad q_1=1/32,\quad q_2=-1/32.$$

Hence

$$(X_1'X_1)^{-1}=\frac{1}{32}(5B_0+B_1-B_2).$$

Therefore, the design T is a Partially Balanced Resolution Ⅳ* design with parameters ($m=10$, $n=20$; $\mu_0=5/64$, $\mu_1=1/64$, $\mu_2=-1/64$).

In the design,

$$tr(V_T)=10\left(\frac{1}{2}q_0\right))=50/64,$$

while, $\tau(m,n)=10/20$.

So,

absolute (trace) efficiency$=10\times 54/20\times 50=0.64$.

Appendix

Consider a PBIB design with parameters $(v, b, r, k;\ \lambda_1, \lambda_2, \cdots, \lambda_t)$, and let n_i, $i=1,2,\cdots,t$, and p_i^{jk}, $i,j,k=1,2,\cdots,t$, be the parameters of its association scheme.

Let

(a. 1) $$B_i=\begin{pmatrix} b^i_{11} & b^i_{12} & \cdots & b^i_{iv}\\ b^i_{21} & b^i_{22} & \cdots & b^i_{2v}\\ & \cdots\cdots & & \\ b^i_{v1} & b^i_{v2} & \cdots & b^i_{vv}\end{pmatrix},\quad i=1,2,\cdots,t,$$

be the association matrices, then we observe that B_i, $i=1,2,\cdots,t$, are symmetrical matrces, with row and column totals eqiual to n_i. In addition, let every treatment be the 0 th associate of itself and no other treatments. Then we see that

(a. 2) $$B_0=I_v,\ n_0=1,\ p^0_{ij}=\begin{cases}n_i, & \text{if } i=j,\\ 0, & \text{if } i\neq j,\end{cases}\ p^i_{0k}=\begin{cases}1, & \text{if } i=k,\\ 0, & \text{if } i\neq k,\end{cases}\ \text{and } \lambda_0=r.$$

Let $N(v\times b)$ be the incidence matrix of the above PBIB design, then it is well known that

(a. 3) $$NN'=rB_0+\lambda_1B_1+\cdots+\lambda_tB_t.$$

Given two treatments α and β, they are either 0 th, 1st,$\cdots$, or t th associates, and hence only one of the elements $b^0_{\alpha\beta}$, $b^1_{\alpha\beta}, \cdots$, $b^t_{\alpha\beta}$ is unity. Hence

(a.4) $$B_0+B_1++B_t=J_v.$$

From the same consideration, it follows that

$$c_0B_0+c_1B_1+\cdots+c_tB_t=0$$

holds if and only if

$$c_0=c_1=\cdots=c_t=0;$$

hence the linear function of $B_0, B_1, \cdots, B_t$ form a vector space of $(t+1)$- demensionality with basis $B_0, B_1, \cdots, B_t$.

Since the (α, β) th element of B_jB_k can be interpreted as the number of treatments common to the j th associates of α and the k th associates of β, we have

(a.5) $$B_jB_k=\sum_{i=0}^{t} p^i_{jk}\, B_i, \quad j,k=0, 1, \cdots, t,$$

and hence the multiplication is closed in the set of linear function of B_0, $B_1, \cdots, B_t$. Clearly this set of linear functions of $B_0, B_1, \cdots, B_t$ forms a commutative group.

Since matrix multiplication is associative

(a.6) $$B_i(B_jB_k)=\sum_u \sum_s p^u_{jk}\, p^s_{iu}\, B_s$$
$$=(B_iB_j)B_k=\sum_u \sum_w p^u_{jk}\, p^w_{uk}\, B_w.$$

The independence of $B_0, B_1, \cdots, B_t$ implies that

(a.7) $$\sum_u p^u_{jk}\, p^r_{iu}=\sum_u p^u_{ij}\, p^r_{uk} \text{ for } r=0, 1, \cdots, t.$$

Now let us define $\mathscr{P}_i$-matrices by

(a.8) $$\mathscr{P}_i=\begin{pmatrix} p^0_{0i} & p^1_{0i} & \cdots & p^t_{0i} \\ p^0_{1i} & p^1_{1i} & \cdots & p^t_{1i} \\ & \cdots & & \\ p^0_{ti} & p^1_{ti} & \cdots & p^t_{ti} \end{pmatrix}, \quad i=0, 1, \cdots, t.$$

In terms of $\mathscr{P}_j$-matrices, (a.7) implies that

(a.9) $$\mathscr{P}_j\mathscr{P}_k=\sum_{i=0}^{t} p^i_{jk}\, \mathscr{P}_i$$

Thus the $\mathscr{P}$-matrices multiply in the same manner as the B-matrices. It can be easily verified that $\mathscr{P}_0$, $\mathscr{P}_1, \cdots, \mathscr{P}_t$ are linearly independent and they form the basis for a vector space and combine in the same way as the B-matrices as well as in multiplication.

Bose and Mesner (1959) proved the following Theorem a.1.

Theorem a.1. The distinct characteristic roots of

$$B=c_0B_0+c_1B_1+\cdots+c_tB_t$$

and the distinct characteristic roots of

$$\mathscr{P}=c_0\mathscr{P}_0+c_1\mathscr{P}_1+\cdots+c_t\mathscr{P}_t$$

are the same.

Proof: Let $f(\lambda)$ and $g(\lambda)$ be the minimal polynomial of B and $\mathscr{P}$, respectively. They are monic polynomials. The term $f(B)$ can be expressed as

$$f(B)=\sum_{i=0}^{t} d_iB_i,$$

whose representation in $(t+1)\times(t+1)$ matriceswill be

$$f(\mathscr{P})=\sum_{i=0}^{t} d_i\mathscr{P}_i.$$

Since $f(\lambda)$ is the minimal polynomial of B, we have $f(B)=0$, which implies that $d_0=d_1=\cdots d_t=0$ and hence $f(\mathscr{P})=0$. Since $f(\mathscr{P})=0$ and since $g(\lambda)$ is the minimal polynomial of $\mathscr{P}$, we have the result that $g(\lambda)$ divides $f(\lambda)$. We can similarly show that $f(\lambda)$ divides $g(\lambda)$. Since both $f(\lambda)$ and $g(\lambda)$ are monic polynomials, $f(\lambda)=g(\lambda)$. Thus B and $\mathscr{P}$ have the same distinct characteristic roots.

Now we can prove the following Theorem a.2:

Theorem a.2. Let $(\alpha NN'+\beta J_v)$ be a nonsingular matrix, then the inverse of the matrix has the form:

(a.10) $$q_0B_0+q_1B_1+\cdots+q_tB_t$$

for real numbers q_i's.

Proof: Since $B_0+B_1+\cdots+B_t=J_v$,

(a.11) $$\alpha NN'+\beta J_v=c_0B_0+c_1B_1+c_2B_2+\cdots+c_tB_t,$$

where $c_0=\alpha r+\beta$, and $c_i=\alpha\lambda_i+\beta$ for $i=1, 2, \cdots, t$.

Suppose that the inverse of $(\alpha NN'+\beta J_v)$ has the following from

$$q_0B_0+q_1B_1+\cdots+q_tB_t,$$

then

$$\begin{aligned}I_v=B_0&=(\sum_{i=0}^{t}q_iB_i)\ (\sum_{i=0}^{t}c_iB_i)\\&=\sum_j\sum_k q_jc_kB_jB_k\\&=\sum_j\sum_k q_jc_k(\sum_i p_{jk}^i\ B_i)\\&=\sum_j\sum_k q_jc_k\,p_{jk}^0\ B_0+\sum_j\sum_k q_jc_k\ p_{jk}^1B_1+\cdots+\sum_j\sum_k q_jc_k\ p_{jk}^tB_t.\end{aligned}$$

Since B_i's are independent,

$$\sum_j\sum_k q_jc_k\,p_{jk}^0=1$$

$$\sum_j \sum_k q_j c_k p^1_{jk} = 0$$
$$\cdots$$
$$\sum_j \sum_k q_j c_k p^t_{jk} = 0,$$

i.e.,

(a.13)
$$\left(c_0 \begin{pmatrix} p^0_{00} & p^0_{10} \cdots p^0_{t0} \\ p^1_{00} & p^1_{10} \cdots p^1_{t0} \\ & \cdots \\ p^t_{00} & p^t_{10} \cdots p^t_{t0} \end{pmatrix} + c_1 \begin{pmatrix} p^0_{01} & p^0_{11} \cdots p^0_{t1} \\ p^1_{01} & p^1_{11} \cdots p^1_{t1} \\ & \cdots \\ p^t_{01} & p^t_{11} \cdots p^t_{t1} \end{pmatrix} + \cdots + c_t \begin{pmatrix} p^0_{0t} & p^0_{1t} \cdots p^0_{tt} \\ p^1_{0t} & p^1_{1t} \cdots p^1_{tt} \\ & \cdots \\ p^t_{0t} & p^t_{1t} \cdots p^t_{tt} \end{pmatrix} \right) \begin{pmatrix} q_0 \\ q_1 \\ \vdots \\ q_t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Hence, the equation (a.13) can be written as follows:

(a.14)
$$[c_0\mathscr{P}_0' + c_1\mathscr{P}_1' + \cdots + c_t\mathscr{P}_t']\boldsymbol{q} = \boldsymbol{d},$$

where $\boldsymbol{q}' = (q_1, q_1, \cdots q_t)$ and $\boldsymbol{d}' = (1, 0, \cdots 0)$.

From the Theorem a.1, since $\sum_i c_i B_i$ is symmetric nonsingular matrix, the characteristic roots of the matrix $\sum_i c_i \mathscr{P}_i'$ are all real and nonzero. Therefore, the equation (a.14) has the unique solution. This means that there exist the real numbers q_0, $q_1, \cdots, q_t$, such that

$$(\alpha NN' + \beta J_v)^{-1} = q_0 B_0 + q_1 B_1 + \cdots + q_t B_t.$$

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