Selection Problems in terms of Coefficients of Variation

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ABSTRACT

Selection procedures are proposed for selecting the 'best' industrial process with the smallest fraction defective. For normally distributed industrial processes, this is equivalent to selecting in terms of coefficients of variation. For the case of known variances, selection procedures by Bechhofer (1954), and Bechhofer and Turnball (1978) are appropriate. We treat this problem for the case of unknown variances with or without reference to a standard. The large sample solutions of design constants are tabulated and the performance of these approximate solutions are investigated.

1. Introduction

Suppose that we have k industial processes $\Pi_1, \cdots, \Pi_k$ producing similar items, and that the quality of each item produced by Π_i is characterized by a normal random variable X_i with mean μ_i and variance $\sigma_i^2 (i=1, \cdots, k)$. Each item is considered satisfactory if X_i exceeds a given lower specification limit L. Since L may be assumed to be 0, the fraction defective in the process Π_i is then

$$p_i = P_r(X_i < 0) = \Phi(-\mu_i/\sigma_i)$$

where $\Phi(\cdot)$ is the cdf of standard normal distribution. This paper studies selection problems in terms of the fraction defectives or, equivalently, the coeffcients of variation under the framework of the socalled indifference-zone approach of Bechhofer (1954).

* Department of Computer Science and Statistics, Seoul National University. This work was supported in part by the Ministry of Educations, Korean Government, through the Research Institute of Basic Sciences, Seoul National University. The assistance of Mr. Hyung Tae Hwang with computer work is gratefully acknowledged.

For the case of known variances, this is reduced to the selection problem in terms of means. The procedures of Bechhofer (1954), Bechhofer and Turnbull (1977), and Bechhofer and Turnbull (1978) may be appropriate for the comparison of k processes with or without a standard.

For the case of unknown variances, the problem is essentially the selection problem in terms of coefficients of variation. This problem is not considered in the literature so far.

Section 2 treats the problem of selecting the 'best' process with reference to a standard. Because of the close relationship of this problem to the sampling acceptance plan in statistical quality control, we formulate the problem in a manner similar to the sampling acceptance plan. We propose two procedures - one for the case of common unknown variance, and another for the case of unequal unknown variances. The procedures are designed to satisfy the two basic probability requirements. Also, some other properties of the procedures are studied. Computer program has been written to find the design constants, which is available upon request. Large sample solutions of the problem are tabulated, which are compared with the exact solutions for some selected cases.

In Section 3, we consider the problem of selecting the 'best' process among k processes without reference to a standard. We propose a procedure for the case of unequal unknown variances with a modified 'indifference-zone'. The infimum of the probability of seleting the 'best' process is found in order to determine the necessary sample size. A large sample solution of the problem is also derived, and compared with the exact solution for some selected cases.

2. Selection of the best with reference to a standard

We assume that the quality characteristic of each item produced by the process Π_i is normally distributed with unknown mean μ_i and unknown variance $\sigma_i^2 (1 \leq i \leq k)$. Further, it is assumed that there is a lower specification 0.

Let $p_i = \Phi(-\mu_i/\sigma_i)$ denote the fraction defective in the process Π_i $(1 \leq i \leq k)$. The ordered values of the p_i and $\theta_i = \mu_i/\sigma_i$ are denoted by $p_{[1]} \leq \cdots \leq p_{[k]}$ and $\theta_{[1]} \leq \cdots \leq \theta_{[k]}$, respectively.

For a given standard p_1^*, the *goal* of the experiment is to select the 'best' process, i.e., the one associated with the smallest fraction defective $p_{[1]}$ provided $p_{[1]} < p_1^*$, and in case no process has the fraction defective smaller than p_1^*, then to reject all the processes.

As is usually done in the sampling acceptance plan, it is assumed that, prior to experimentation, the experimenter can specify the constants $\{p_0^*, p_1^*, \alpha, \beta\}$ where $0<p_0^*<p_1^*<1, 0<\alpha<1, 0<\beta<1$, so that any selection rule satisfies the following *probability requirements:*

$$P_r\{\text{selecting the best}\}\geq 1-\alpha \tag{2.1a}$$

whenever $p_{[1]}\leq p_0^*$, $p_{[2]}\geq p_1^*$, and

$$P_r\{\text{rejecting all the processes}\}\geq 1-\beta \tag{2.1b}$$

whenever $p_{[1]}\geq p_1^*$

Note that, in this formulation, p_0^* and p_1^* play the role of the acceptable quality level and the lot tolerance percent defective, respectively, in the sampling acceptance plan.

(A) The case of common unknown variance

We propose the following natural selection procedure to guarantee (2.1a) and (2.1b) when $\sigma_1^2=\cdots=\sigma_k^2=\sigma^2$ is unknown:

Take n independent observations $X_{ij}(1\leq j\leq n)$ from Π_i $(1\leq i\leq k)$. Compute $\overline{X}_i=\sum_{j=1}^{n} X_{ij}/n$ and $S^2=\sum_{i=1}^{k}\sum_{j=1}^{n}(X_{ij}-\overline{X}_i)^2/k(n-1)$. If $\max \overline{X}_i\leq cS$, reject all the processes, and if $\max \overline{X}_i>cS$, then select as the best the process yielding the largest sample mean. (2.2)

To implement the procedure, we need the *design constants*, i.e., the sample size n and the appropriate c which guarantee (2.1a) and (2.1b). Let $\Phi(\cdot)$ denote the *cdf* of the standard normal distribution, and let $g_\nu(\cdot)$ denote the *pdf* of chi-squared distribution with ν degrees of freedom. Also, let $\overline{X}_{(j)}/S$ denote the statistic associated with $\theta_{[j]}$, $j=1,\cdots,k$.

Then, for $\nu=k(n-1)$ and $\theta_j^*=-\Phi^{-1}(p_j^*)$ $(j=0,1)$

$$\begin{aligned}
&P_r\ \{\text{selecting the best}\}\\
&=P_r\{\overline{X}_{(k)}/S\geq c,\ \overline{X}_{(k)}/S=\max_{1\leq j\leq k}\overline{X}_{(j)}/S\}\\
&=\int_0^\infty\int_{c\sqrt{ny/\nu}}^\infty \prod_{i=1}^{k-1}\Phi(x-\sqrt{n}\ \theta_{[i]})d\ \Phi(x-\sqrt{n}\theta_{[k]})g_\nu(y)dy\\
&\geq\int_0^\infty\int_{c\sqrt{ny/\nu}}^\infty \Phi^{k-1}(x-\sqrt{n}\ \ \theta_1^*)\ d\Phi(x-\sqrt{n}\ \ \theta_0^*)\ g_\nu(y)dy
\end{aligned}$$

whenever $p_{[1]} \leq p_0^*$ and $p_{[2]} \geq p_1^*$, i.e., $\theta_{[k]} \geq \theta_0^*$ and $\theta_{[k-1]} \leq \theta_1^*$.

Also, P_r {rejecting all the processes}

$$=P_r\{\overline{X}_i \leq cS,\ i=1,\cdots,\ k\}$$

$$=\int_0^\infty \prod_{i=1}^k \Phi(c\sqrt{ny/\nu}-\sqrt{n}\ \theta_i^*)\ g_\nu(y)dy$$

$$\geq \int_0^\infty \Phi^k(c\sqrt{ny/\nu}-\sqrt{n}\ \theta_1^*)\ g_\nu(y)dy$$

whenever $p_{[1]} \geq p_1^*$, i.e., $\theta_{[k]} \leq \theta_1^*$. Thus, we have the next result.

Theorem 2.1. In order to guarantee (2.1a) and (2.1b), the smallest sample size n and c in the procedure (2.2) should be chosen to satisfy

$$\int_0^\infty \int_{c\sqrt{ny/\nu}}^\infty \Phi^{k-1}(x-\sqrt{n}\quad \theta_1^*)d\quad \Phi(x-\sqrt{n}\theta_0^*)\ g_\nu(y)dy \geq 1-\alpha \tag{2.3a}$$

and

$$\int_0^\infty \Phi^k(c\sqrt{ny/\nu}-\sqrt{n}\quad \theta_1^*)\ g_\nu(y)dy \geq 1-\beta \tag{2.3b}$$

where $\nu=k(n-1)$ and $\theta_j^*=-\Phi^{-1}(p_j^*)$ $(j=0,1)$.

Computer program to solve (2.3a) and (2.3b) has been prepared using 16 point Gauss-Laguerre quadrature formula, which is available upon request. For selected values of k, p_0^*, p_1^*, α and β, the program has been run and it has been found to be time consuming. Therefore, it would be useful to have a large sample approximation to the exact solution.

It follows from the asymptotic distribution of $(\overline{X}_1,\cdots,\overline{X}_k,\ S^2)$ that, for large n,

$$\sqrt{n}(\overline{X}_i/S-\theta_i)\sim Z_i+\theta_i Z_0/\sqrt{2k}\ \ (i=1,\cdots,\ k)$$

where $Z_0,\ Z_1,\cdots,\ Z_k$ are independent standard normal random variables. Therefore, for $\theta_1=\cdots=\theta_{k-1}=\theta_1^*$ and $\theta_k=\theta_0^*$,

$$P_r\{\text{selecting the best}\}$$

$$=P_r\{\overline{X}_k/S \geq c,\ \ \overline{X}_k/S=\max\ (X_i/S)\}$$

$$\cong P_r\{Z_k-\theta_0^*\ Z_0/\sqrt{2k} \geq \sqrt{n}(c-\theta^*{}_0),$$

$$Z_k-\theta_0^*\ /\sqrt{2k}\ Z_i-\theta_1^*\ Z_0/\sqrt{2k}+\sqrt{n}(\theta_1^*-\theta_0^*)\ i=1,\cdots,\ k-1\}$$

$$=\int_{-\infty}^\infty \int_{\sqrt{n}(c-\theta_0^*)}^\infty \Phi^{k-1}\{x+\theta_1^*\ y/\sqrt{2k}+\sqrt{n}(\theta_1^*-\theta_0^*)\}$$

$$d\Phi(x+\theta_0^*\ y/\sqrt{2k})d\Phi(y)$$

and, for $\theta_1=\cdots=\theta_{k-1}=\theta_k=\theta_1^*$,

$$P_r\{\text{rejecting all the processes}\}$$
$$=P_r\{\overline{X}_i/S\leq c,\ i=1,\cdots,k\}$$
$$\cong P_r\{Z_i-\theta_1^*\ Z_0/\sqrt{2k}\leq\sqrt{n}(c-\theta_1^*),\ i=1,\cdots,k\}$$
$$=\int_{-\infty}^{\infty}\Phi^k(\theta_1^*\ x/\sqrt{2k}+\sqrt{n}(c-\theta_1^*))d\ \Phi(x)$$

Hence, the large sample solution of (2.3a) and (2.3b) is given by

$$n=[\{g/(\theta_0^*-\theta_1^*)\}^2]+1,\ \ c=\theta_1^*+h(\theta_0^*-\theta_1^*)/g \qquad (2.4)$$

Where the constants g and h satisfy

$$\int_{-\infty}^{\infty}\int_{h-g}^{\infty}\Phi^{k-1}(x+\theta_1^*y/\sqrt{2k}+g)\ \ d\Phi(x+\theta_0^*\ y/\sqrt{2k})\ d\Phi(y)$$
$$=1-\alpha \qquad (2.5a)$$

$$\int_{-\infty}^{\infty}\Phi^k(\theta_1^*\ x/\sqrt{2k}+h)\ d\Phi(x)=1-\beta \qquad (2.5b)$$

and $[\cdot]$ denotes the greatest integer function. The values of n and c given by (2.4) are given in Table 1 for $k=2(1)5$, $1-\alpha=0.95$, $1-\beta=0.9$ and selected values of p_0^* and p_1^*. These values are obtained by the large sample approximation. Hence, to see how accurate these approximations are, we have compared the values in Table 1 with the exact solution for the following cases:

%		$k=2$			
		Exact		Approximate	
p^*_0	p^*_1	n	c	n	c
6	18	37	1.2308	37	1.2091
	24	20	1.1129	21	1.0886
	36	10	0.9403	10	0.8848
8	24	29	1.0503	29	1.0272
	32	16	0.9224	16	0.8907
	48	7	0.7218	8	0.6520
10	30	23	0.9004	24	0.8725
	40	12	0.7653	13	0.7194
	60	6	0.4414	6	0.4403

The computations were done using 16 point Gauss-Hermite quadrature for the approximate solutions, and using 16 point Gauss-Laquerre quadrature for the exact solutions, both in the IBM Scientific Subroutine Package. As can be seen from the comparison results, the approximations to n and c are sufficiently accurate enough for practical purposes. Furthermore, the comparison result shows that the approximation to n is quite good even for small n; hence we can use the values of n and c in Table 1 as an initial

Table 1. **Large sample solutions for the design constants (n, c) in the procedure (2.2) for $1-\alpha=0.95$, $1-\beta=0.9$.**

%		$k=2$		$k=3$		$k=4$		$k=5$	
p_0^*	p_1^*	n	c	n	c	n	c	n	c
1	3	112	2.0872	103	2.1016	99	2.1103	97	2.1174
	4	65	2.0132	60	2.0327	58	2.0452	57	2.0540
	6	35	1.8982	33	1.9264	32	1.9443	31	1.9569
2	6	78	1.7853	74	1.8019	72	1.8124	72	1.8198
	8	45	1.6994	43	1.7224	42	1.7368	42	1.7469
	12	24	1.5632	23	1.5970	23	1.6181	23	1.6328
4	12	50	1.4399	50	1.4599	50	1.4721	50	1.4807
	16	29	1.3358	28	1.3638	29	1.3809	29	1.3928
	24	15	1.1649	15	1.2071	15	1.2328	15	1.2505
6	18	37	1.2091	38	1.2316	38	1.2454	39	1.2549
	24	21	1.0886	21	1.1207	22	1.1401	22	1.1535
	36	10	0.8848	11	0.9337	11	0.9631	11	0.9832
8	24	29	1.0272	30	1.0521	31	1.0671	32	1.0775
	32	16	0.8907	17	0.9263	17	0.9477	18	0.9624
	48	8	0.6520	8	0.7062	8	0.7386	9	0.7608
10	30	24	0.8725	25	0.8995	26	0.9158	27	0.9270
	40	13	0.7194	14	0.7581	14	0.7813	15	0.7971
	60	6	0.4403	6	0.4980	7	0.5329	7	0.5569

quess if we want to search for the exact solution.

The performance characteristics of the procedure(2.2) are given in the next result when changes are made in the indifference-zones for this problem.

Theorem 2. 2. The procedure (2.2) with n and c as in Theorem 2·1 also guarantees, for $2\leq t\leq k$,

$$P_r\{\text{selecting any one of the } t \text{ best}\}\geq 1-\alpha \qquad (2.6)$$

whenever $p_{[1]}\leq p_0^*$ and $p_{[t]}\leq p_1^*\leq p_{[t+1]}$.

Proof. It is easy to see that

$$P_r\{\text{selecting any one of the } t \text{ best}\}$$
$$=P_r\{\max_{k-t+1\leq j\leq k} \overline{X}_{(j)}/S\geq c,\ \max_{k-t+1\leq j\leq k} \overline{X}_{(j)}/S\geq \max_{1\leq j\leq k-t} \overline{X}_{(j)}/S\} \quad (2.7)$$

where $\overline{X}_{(j)}/S$ is associated with $\theta_{[j]}, j=1, \cdots, k$. Note that, given $S/\sigma=w$, $\overline{X}_{(1)}, \cdots, \overline{X}_{(k)}$ are independent and normally distributed with mean $\theta_{[j]}/w$ $(j=1, \cdots, k)$ and variance $1/(nw^2)$, respectively. Hence, given $S/\sigma=w$, the distribution of $\overline{X}_{(j)}$ is stochastically increasing in $\theta_{[j]}$. Therefore, (2.7) is increasing in $\theta_{[j]}$ for $k-t+1\leq j\leq k$ and decreasing in $\theta_{[j]}$ for $1\leq j\leq t$, i.e., decreasing in $p_{[j]}$ for $1\leq j\leq t$ and increasing in $p_{[j]}$ for $t+1\leq j\leq k$.

Thus,

$$P_r\{\text{selecting any one of the } t \text{ best} \mid p_{[1]}\leq p_0^*,\ p_{[t]}\leq p_1^*\leq p_{[t+1]}\}$$
$$\geq P_r\{\text{selecting any one of the } t \text{ best} \mid p_{[1]}=p_0^*,\ p_{[2]}=\cdots=p_{[k]}=p_1^*\}$$
$$\geq P_r\{\text{selecting the best} \mid p_{[1]}=p_0^*,\ p_{[2]}=\cdots=p_{[k]}=p_1^*\}$$
$$\geq 1-\alpha,$$

which completes the proof.

(B) The case of unequal unknown variances

When $\sigma_1^2,\cdots,\sigma_k^2$ are unknown, the following natural selection procedure can be used to guarantee (2.1a) and (2.1b):

Take n independent observations X_{ij} $(1\leq j\leq n)$ from Π_i $(1\leq i\leq k)$.

Compute $T_i=\overline{X}_i/S_i$ where $S_i^2=\sum_{j=1}^{n}(X_{ij}-\overline{X}_i)^2/(n-1)$.

If $\max T_i\leq c$, reject all the processes, and if $\max T_i>c$, then select as the best the process yielding the largest T_i. (2.8)

The design constants n and c to implement the procedure (2.8) should be determined by the next result, which can be proved in a way similar to the proof of Theorem 2.1.

Theorem 2.3. In order to grarantee (2.1a) and (2.1b), the smallest sample size n and c in the procedure (2.8) should be chosen to satisfy

$$\int_{c\sqrt{n}}^{\infty} F^{k-1}(x \mid \sqrt{n}\theta_1^*)\,dF(x\mid \sqrt{n}\theta_0^*)\geq 1-\alpha \tag{2.9a}$$

$$F^k(\sqrt{n}c \mid \sqrt{n}\theta_1^*)\geq 1-\beta \tag{2.9b}$$

where $F(x|\lambda)$ denotes the *cdf* of non-central t distribution with $n-1$ degrees of freedom and the non-centrality parameter λ.

A computer program to evaluate the left-hand sides of (2.9a) and (2.9b) has been prepared using 16 point Gauss-Laguerre quadrature formula.

As in the case A, the large sample solution of (2.9a) and (2.9b) follows from the fact that, for large n,

$$\sinh^{-1}(T_i/\sqrt{2})\sim N(\sinh^{-1}(\theta_i/\sqrt{2}),\ 1/2n).$$

Suppose that h and g satisfy the simultaneous equations

$$\int_{h-g}^{\infty} \Phi^{k-1}(x+g)d\Phi(x)=1-\alpha \qquad (2.10a)$$

$$\Phi^{k}(h)=1-\beta. \qquad (2.10b)$$

Then, the large sample solution of (2.9a) and (2.9b) is given by

$$n=\left[\frac{1}{2}\{g/(\xi_0-\xi_1)\}^2\right]+1, c=\sqrt{2}\ \sinh\{\xi_1+h(\xi_0-\xi_1)/g\} \qquad (2.11)$$

where $\xi_1=\sinh^{-1}(\theta_1^*/\sqrt{2})$ and $\xi_0=\sinh^{-1}(\theta_0^*/\sqrt{2})$.

The values of n and c given by (2.11) are given in Table2 for $k=2(1)5$, $1-\alpha=0.95$, $1-\beta=0.9$ and selected values of p^*_0 and p^*_1.

Table 2. **Large sample solutions for the design constants ($n.c$) in the procedure (2.8) for $1-\alpha=0.95$, $1-\beta=0.90$.**

%		$k=2$		$k=3$		$k=4$		$k=5$	
p^*_0	p^*_1	n	c	n	c	n	c	n	c
1	3	178	2.0926	199	2.1040	214	2.1112	225	2.1164
	4	102	2.0209	114	2.0358	123	2.0451	129	2.0517
	6	53	1.9102	60	1.9300	64	1.9425	67	1.9514
2	6	116	1.7903	130	1.8031	140	1.8112	147	1.8169
	8	65	1.7068	73	1.7234	78	1.7339	83	1.7414
	12	33	1.5746	37	1.5971	40	1.6113	42	1.6214
4	12	69	1.4442	77	1.4590	83	1.4683	87	1.4750
	16	38	1.3421	42	1.3615	45	1.3737	47	1.3824
	24	18	1.1743	20	1.2009	22	1.2177	23	1.2297
6	18	48	1.2127	53	1.2291	57	1.2394	60	1.2468
	24	25	1.0938	28	1.1155	30	1.1292	32	1.1390
	36	12	0.8921	13	0.9224	14	0.9415	15	0.9551
8	24	35	1.0301	40	1.0480	43	1.0593	45	1.0673
	32	19	0.8948	21	0.9187	22	0.9337	23	0.9445
	48	8	0.6579	9	0.6918	10	0.7132	10	0.7285
10	30	28	0.8747	31	0.8941	33	0.9063	35	0.9150
	40	14	0.7226	16	0.7487	17	0.7651	18	0.7768
	60	6	0.4461	7	0.4841	7	0.5080	8	0.5251

To see how accruate these apporximate solutions are, we have computed the actual values of (2.9a)and (2.9b) for $k=2$; the values below are the actual values for nominal $1-\alpha=0.95$, $1-\beta=0.90$.

As can be seen from the below computation, the actual values of $1-\alpha$ $(1-\beta)$ are

p^*_0	p^*_1	$1-\alpha$	$1-\beta$	p^*_0	p^*_1	$1-\alpha$	$1-\beta$	p^*_0	p^*_1	$1-\alpha$	$1-\beta$
6	18	0.940	0.922	8	24	0.941	0.926	10	30	0.937	0.919
	24	0.939	0.932		32	0.934	0.923		40	0.936	0.926
	36	0.931	0.929		48	0.934	0.931		60	0.930	0.912

slightly less (larger) than the nominal values, where the differences are small enough for practical purposes. The values of n and c in Table 2 can be also used as an initial guess if one wishes to search for the exact solution for particular p_0^*, p_1^*, k. It should also be remarked that the values of h and g satisfying (2.10a) and (2.10b) are tabulated by Bechhofer and Turnbull (1978) for some other values of α and β and $k=2(1)5$.

The following performance characteristic of the procedure (2.8) can be obtained in the same way as Theorem 2.2 was proved except that we need the stochastic ordering property of the non-central t distribution.

Theorem 2. 4. The procedure (2.8) with n and c as in Theorem 2 also guarantees, for $2\leq t\leq k$,

$$P_r[\text{selecting any one of the } t \text{ best}]\geq 1-\alpha \qquad (2.12)$$

whenever $p_{[1]}\leq p_0^*$ and $p_{[t]}\leq p_1^*\leq p_{[t+1]}$.

3. Selection of the best without a standard

As in the previous section, we assume that the quality of the process Π_i is characterized by normal distribution with unknown mean μ_i and unknown variance σ^2_i $(1\leq i\leq k)$. Also, it is assumed that there is a lower specification limit 0.

In this section, we consider the problem of selecting the 'best' process without reference to a standard, where the 'best' process is clealy associated with $\theta_{[k]}$

Follwing the indifference-zone approach by Bechhofer (1954), the experimenter, prior to the experimentation, specifies two constants $\Delta^*>0$ and $\alpha(1/k<1-\alpha<1)$, which are incorporated into a probability requirement

$$P_r\{\text{selecting the best}\}\geq 1-\alpha \qquad (3.1)$$

whenever $\theta_{[k]}\geq\theta_{[k-1]}+\Delta^*$.

For this purpose, it is natural to consider a selection procedure based on a statistic $T_i=\overline{X}_i/S_i$. However, it can be easily shown that the minimum probability requirement (3.1) can not be satisfied by any selection procedure based on T_i $(1\leq i\leq k)$ (see, for example, Dudewicz (1971)). On the other hand, the experimenter assumes that $a \leq \theta_i \leq b$ $(1 \leq i \leq k)$ in many practical situations. Thus, with such a restricted parameter space, the minimum probability requirement is modefied as follows:

$$P_r\{\text{selecting the best}\} \geq 1-\alpha \qquad (3.2)$$

whenever $\theta_{[k]} \geq \theta_{[k-1]} + \Delta^*$ and $a \leq \theta_i \leq b$ $(1 \leq i \leq k)$.

we propose the following natural selection procedure to satisfy the probability requirement (3.2).

Take n independent observations $X_{ij} (1 \leq j \leq n)$ from Π_i $(1 \leq i \leq k)$. Compute $T_i = \bar{X}_i / S_i$. Then, select as the best the process yielding the max T_i. (3.3)

Note that the above selection procedure is equivalent to that in terms of $\hat{\theta}_i = h(T_i)$ for a non-decreasing function h. In this respect, we remark that any Bayes estimator of θ_i with respect to squared error loss is a non-decreasing function of T_i.

In the sequel, we study the probability of selecting the best as a function of θ_i to get the minimum sample size n which guarantees (3.2). For this purpose, let $F(x|\lambda) = F_{n-1}(x|\lambda)$ denote the *cdf* of non-central t distribution with $n-1$ degrees of freedom and non-centrality parameter λ.

Lemma 3.1. The procedure (3.3) satisfies the following inequality.

$$P_r\{\text{selecting the best} | \theta_{[k]} \geq \theta_{[k-1]} + \Delta^*, \ a \leq \theta i \leq b \ i=1, \cdots, k\}$$
$$\geq \inf \{ \int_{-\infty}^{\infty} F^{k-1}(x|\lambda) \, dF(x|\lambda+\Delta) \mid \sqrt{n}a \leq \lambda \leq \sqrt{n}(b-\delta^*) \}$$

where $\Delta = \sqrt{n}\Delta^*$

Proof. The result follows from the stochastic ordering property of $F(x|\lambda)$ in λ.

Lemma 3.2. Suppose that $H(x, \lambda)$ satisfies the following;

(1) For fixed λ, $H(x, \lambda) \leq 0$ for $x > 0$ and $H(x, \lambda) \geq 0$ for $x < 0$,

(2) $H(x, \lambda)$ is non-increasing in λ for fixed x.

Then, there exists λ_0 such that

$$\int_{-\infty}^{\infty} F^{k-2}(x|\lambda) \ H(x, \lambda) dx \geq 0 \qquad \text{for } \lambda < \lambda_0,$$

and

$$\int_{-\infty}^{\infty} F^{k-2}(X|\lambda) \ H(x, \lambda) dx \leq 0 \qquad \text{for } \lambda > \lambda_0,$$

unless the left sides of the inequalities are either positive for all λ or negative for all λ.

Proof. Since the *pdf* of non-central distribution has the monotone likelihood ratio property in λ, $F(x|\lambda_2)/F(x|\lambda_1)$ is non-decreasing in x for $\lambda_1 < \lambda_2$. Thus, it follows from the properties of $H(x,\lambda)$ that, for $\lambda_1<\lambda_2$,

$$\int_{-\infty}^{\infty} F^{k-2}(x|\lambda_2)\ H(x;\lambda_2)dx$$

$$=(\int_{-\infty}^{0}+\int_{0}^{\infty})\{F(x|\lambda_2)/F(x|\lambda_1)\}^{k-2}F^{k-2}(x|\lambda_1)\ H(x,\lambda_2)dx$$

$$\leq c^{k-2}\int_{-\infty}^{\infty} F^{k-2}(x|\lambda_1)\ H(x,\lambda_2)dx$$

$$\leq c^{k-2}\int_{-\infty}^{\infty} F^{k-2}(x|\lambda_1)\ H(x,\lambda_1)dx$$

where $c=F(0|\lambda_2)/F(0|\lambda_1)>0$.

Thus, for $\lambda_1<\lambda_2$,

$$\int_{-\infty}^{\infty} F^{k-2}(x|\lambda_1)\ H(x,\lambda_1)dx<0$$

implies

$$\int_{-\infty}^{\infty} F^{k-2}(x|\lambda_2)\ H(x,\lambda_2)dx<0$$

Therefore, the result now follows by taking

$$\lambda_0=\inf\{\lambda;\int_{-\infty}^{\infty} F^{k-2}(x|\lambda)\ H(x,\lambda)dx<0\}.$$

Lemma 3.3. Let

$$I(\lambda)=\int_{-\infty}^{\infty} F^{k-1}\ (x|\lambda)\ dF\ (x|\lambda+\Delta).$$

Then, $\frac{d}{d\lambda}I(\lambda)$ changes the sign exactly once from + to − as λ varies from $-\infty$ to ∞ ; In particular, $\frac{d}{d\lambda}\ I(\lambda)<0$ for $\lambda>0$.

Proof. Denoting the *pdf* of $F(x|\lambda)$ by $f(x|\lambda)$, we have

$$\frac{d}{d\lambda}I(\lambda)=(k-1)\int_{-\infty}^{\infty} F^{k-2}(x|\lambda)\{f(x|\lambda+\Delta)\frac{d}{d\lambda}F(x|\lambda)-f(x|\lambda)$$
$$\frac{d}{d\lambda}F(x|\lambda+\Delta)\}dx$$

$$=c\int_{-\infty}^{\infty} F^{k-2}(x|\lambda)\ H(x,\lambda)dx/\exp(\lambda^2+\lambda\Delta)$$

where $c=c(n,k,\Delta)$ does not depend on λ and

$$H(x,\lambda)=\iint_{u\leq v} g(u,v)\ (\sqrt{v}-\sqrt{u})\ (e^{\Delta x\sqrt{u/(n-1)}}-e^{\Delta x\sqrt{v/(n-1)}})\ du\ dv$$

with

$$g(u,v)=(\sqrt{uv})^{n-3}\ e^{-x^2(u+v)/2(n-1)}\ e^{\lambda x(\sqrt{u}+\sqrt{v})/\sqrt{n-1}}$$

Since $H(x,\lambda)$ satisfies the assumptions of Lemma 3.2, $\frac{d}{d\lambda}\ I(\lambda)$ changes sign at most once from + to −: Furthermore, it can be easily shown that

$$H(x,0)\leq -H(-x,0)<0 \qquad \text{for } x>0.$$

Hence,

$$\left[\frac{d}{d\lambda} I(\lambda)\right]_{\lambda=0} = c\int_0^\infty \{F^{k-2}(x|0)\ H(x,0) + F^{k-2}(-x|0)\ H(-x,0)\}\,dx$$

$$\leq c\int_0^\infty \{F^{k-2}(-x|0) - F^{k-2}(x|0)\}\ H(-x,0)dx$$

$$<0$$

which implies the result.

The next result follows from Lemma's 3. 1, 3. 2 and 3. 3.

Theorem 3. 1. In order to guarantee (3. 2), the smallest sample size n in the procedure (3. 3) should be chosen to satisfy

$$\min\{\int_{-\infty}^{\infty} F^{k-1}(x|\sqrt{n}a)dF(x|\sqrt{n}a+\sqrt{n}\Delta^*),$$

$$\int_{-\infty}^{\infty} F^{k-1}(x|\sqrt{n}(b-\Delta^*))dF(x|\sqrt{n}b)\}\geq 1-\alpha. \tag{3. 4}$$

In many practical applications, a is likely to be positive, i.e., the associated fraction. defective is less than 0. 5. In such a case, the minimum of the left side of (3. 4) is the second member by Lemma 3. 3. Hence, in practice, we only need to find n so that

$$\int_{-\infty}^{\infty} F^{k-1}(x|\sqrt{n}(b-\Delta^*))\ dF(x|\sqrt{n}b)\geq 1-\alpha. \tag{3. 5}$$

For simplity, we consider only the case $a>0$ in the remainder of the discussion.

As in Section 2, the large sample solution of (3. 5) easily follows from the fact that, for large n,

$$\sinh^{-1}(T_i/\sqrt{2}) \sim N(\sinh^{-1}(\theta_i/\sqrt{2}),\ 1/2n).$$

In fact, the large sample solution of (3. 5) is given by

$$n=\left[\frac{d^2}{2}\left\{ln\left(\frac{b+\sqrt{2+b^2}}{b-\Delta^*+\sqrt{2+(b-\Delta^*)^2}}\right)\right\}^2\right]+1 \tag{3. 6}$$

where d is the solution of

$$\int_{-\infty}^{\infty} \Phi^{k-1}(x+d)\ d\Phi(x)=1-\alpha$$

The values of d can be found in Gupta, Nagel and Panchapakesan (1973) for selected values of k and α. Also, we have wirtten a computer program to evaluate the left side of (3. 5) which can be used to find the exact solution. We have run this program to see the accuracy of the approximate solutions given by (3. 6). For example, consider the case with $k=2$, $b=2$, $\Delta^*=0.6$, 0.8 and $a>0$. The follwing result shows the values of n by (3. 6) for nominal $1-\alpha=0.90$, 0.95 with the actual values of $1-\alpha$ computed by our program. The values of n are moderately large, and it shows that the approximate

Δ*	n	nominal	actual	Δ*	n	nominal	actual
0.6	37	0.95	0.948	0.8	20	0.95	0.949
	23	0.90	0.898		12	0.90	0.894

solutions are sufficiently accurate for practical purposes for moderately large n.

(Received May 1982; Revised October 1982)

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