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—Research Paper—

On the Euclidean Center Problem

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Abstract

This paper presents an efficient algorithm for finding a new facility(center) in the Euclidean plane in accordance with minimax criterion: that is, the facility is located to minimize the maximum weighted Euclidean distance. The method given in this paper involves computational geometry. Some possible extensions of this problem are also discussed.

1. INTRODUCTION

Let $P=\{p_1,p_2,\ldots,p_n\}$ be a given set of points in the Euclidean plane E_2 representing the locations of n existing facilities. Let p be a point in the Euclidean plane E_2 to be determined as the site of the new facility so as to minimize an appropriately defined cost function $f(p)$. The weight ω_i attached to the existing facility p_i is a given non-negative number which represents interaction between the existing facility p_i and the new facility p. If we define a cost function $f^M(p)$, as the sum of the weighted distances, $\sum_{i=1}^{n}\omega_i d(p_i,p)$, then finding the location of p which minimizes $f^M(p)$ is referred to as the general Fermat problem. This problem has received considerable attention and several approaches (Kuhn's modified gradient [1], hyper-boloid approximation proccdure [2], etc.) have been developed. Unfortunately, it has been shown that for $n\geq 5$ this problem is in general not solvable by ruler-and-compass geometrical constructions [3].

On the other hand, if we define a cost function $f^c(p)$ as the maximum cost, $\max_{1\leq i\leq n}\{\omega_i d(p_i,p)\}$, then finding the location of p which minimzes $f^c(p)$ becomes the minimax version of the general Fermat problem and we call this the Euclidean Center problem(or the minimax single facility location problem). This problem arises when it is more important to provide emergency (or user-oriented) services than long term total service.

In a special case when $\omega_i=1$ for all $1\leq i\leq n$, this problem has a useful geometrical interp-

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retation *i. e.* "find the center of the smallest circle which encloses all the points P". Elzinga and Hearn [4] presented an $0(n^3)$ algorithm which, at each step, constructs a new circle whose diameter is strictly greater than that of the previous circle, and Shamos and Hoey [5] have developed an $0(n \log n)$ algorithm using the farthest Voronoi diagram. But in general case (involving weights other than 1), no one has made use of the special geometrical structure for this problem [6].

In the next section, we introduce the weighted farthest Voronoi diagram which will be used as a main tool in solving geometric problems.

2. THE WEIGHTED FARTHEST VORONOI DIAGRAM

Assume we are given a set $P=\{p_1,p_2,\dots,p_n\}$ of n points in the two-dimensional Euclidean plane E_2 with coordinates $p_i=(a_i,b_i)$ and the non-negative weight ω_i associated with p_i. We define the weighted farthest Voronoi region (WFVR) $\hat{R}_i$ associated with p_i, to be the region having the property that the weighted distance from any $x \in R_i$ to p_i is not less than the weighted distance to any other point $p_j \in P-\{p_i\}$. More precisely,

$$\hat{R}_i=\{x \in E_2 \mid \omega_i d(p_i,x) \geq \max_{p_j \in P} \{\omega_j d(p_j,x)\}$$

where $d(x,y)$ denotes the Euclidean distance between any two points x and y in the plane, Consider any two points $p_i,p_j \in P$. The locus of points to which the weighted distance from p_i is equal to that from p_j is either the perpendicular bisector of p_i and p_j, $h(p_i,p_j)$ (when $\omega_i=\omega_j$) or the circle, $c(p_i,p_j)$, whose center c_{ij} is at

$$\left(\frac{a_i\omega_i^2-a_j\omega_j^2}{\omega_i^2-\omega_j^2}, \frac{b_i\omega_i^2-b_j\omega_j^2}{\omega_i^2-\omega_j^2}\right) \text{ with radius } r_{ij}=\frac{\omega_i\omega_j d(p_i,p_j)}{|\omega_i^2-\omega_j^2|}$$

(when $\omega_i \neq \omega_j$).

Furthermore, if we define $\hat{R}_{ij}=\{x \in E_2 \mid \omega_i d(p_i,x) \geq \omega_j d(p_j,x)\}$, then we have $\hat{R}_i=\bigcap_{j \neq i} \hat{R}_{ij}$. It can be seen that

$$\hat{R}_{ij}=\begin{cases} \text{inside the circle } c(p_i,p_j) \text{ if } \omega_i<\omega_j \\ \text{outside the circle } c(p_i,p_j) \text{ if } \omega_i>\omega_j \\ \text{half plane containing } p_i \text{ defined by } h(p_i,p_j) \text{ if } \omega_i=\omega_j \end{cases}$$

It is clear that $\bigcup \hat{R}_i=E_2$ since any point in the plane must belong to some $\hat{R}_i$. We call the union of the boundaries of $\hat{R}_i$'s (or $\bigcup_{i \neq j} (\hat{R}_i \cap \hat{R}_j)$) the weighted farthest Voronoidiagram (WFVD), denoted by WFVD(P), which is composed of pieces of some straight lines ($h(p_i,p_j)$) and/or circles ($c(p_i,p_j)$). An edge of WFVD is called a Voronoi line, denoted by $l(i,j)$, which represents a part of $c(p_i,p_j)$ (or $h(p_i,p_j)$) and has a property that for any point $x \in l(i,j)$ and $p_k \in P-\{p_i,p_j\}$, $\omega_i d(p_i,x)=\omega_j d(p_j,x) \geq \omega_k d(p_k,x)$ with equality if and only if $x \in l(i,k)$ and $x \in l(j,k)$. An intersection of Voronoi lines $l(i,j)$, $l(i,k)$ is called a Voronoi point and denoted by v_{ijk}. Since v_{ijk} has the property that $\omega_i d(p_i,v_{ijk})=\omega_j d(p_j,v_{ijk})=\omega_k d(p_k,v_{ijk})$, it can be easily seen that $v_{ijk} \in l(j,k)$. Therefore, we can say that every Vornoi point is incident on at least three Voronoi lines. (If a Voronoi point is incident on more than three Voronoi lines, we will regard it as a multiple point). Note that it is possible that $l(i,j)$ and $l(i,k)$ meets at two points. An example of WFVD is shown in Figure 1.

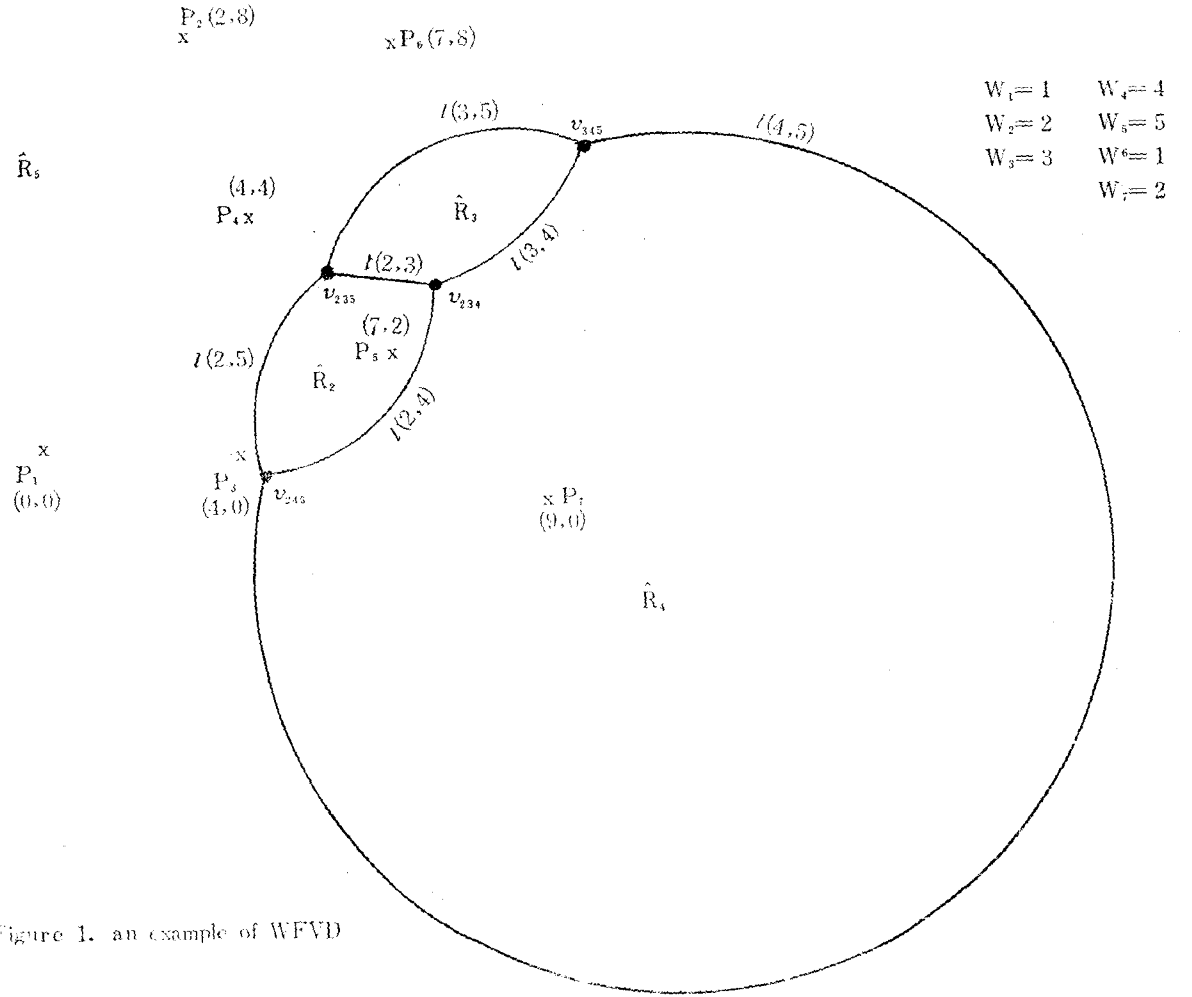

Figure 1. an example of WFVD

3. ALGORITHM

Now we are going to show that the solution of the Euclidean center problem can be easily obtained from the WFVD.

Lemma. Assume the WFVD for a given set of points, $P=\{p_1,p_2,\ldots,p_n\}$, is available. Let p_{ij} be the internal division point which is defined by the intersection point between the straight line segment $\overline{p_ip_j}$ and $c(p_i,p_j)$(or $h(p_i,p_j)$). Then, there exist at most one internal division point on the corresponding Voronoi line $l(i,j)$ in the WFVD. In fact, if one exists, then it is the optimum location point (Eucildean center).

Proof. Suppose there exist two internal division points p_{ij},p_{kl} with $i\neq k$, that is, $p_{ij}\in l(i,j)$, $p_{kl}\in l(k,l)$ and $p_{ij}\neq p_{kl}$.

This implies that $w_id(p_i,p_{ij})=w_jd(p_j,p_{ij})>\max\{w_kd(p_k,p_{ij}),w_ld(p,p_{ij})\}$ (1)

and $w_kd(p_k,p_{kl})=w_ld(p_l,p_{kl})>\max\{w_id(p_i,p_{kl}),w_j,p_{kl})\}$ (2)

Let $p_{ij}\in\hat{R}_{\alpha_1\alpha_2}$ and $p_{kl}\in\hat{R}_{\beta_1\beta_2}$, where $\alpha_1\alpha_2\in\{kl,lk\}$ and $\beta_1\beta_2\in\{ij,ji\}$.

Then we have $w_{\alpha_1}\ d(p_{\alpha_1},\ p_{kl})<w_{\alpha_1}d(p_j,\ p_{ij})$ (3)

and $w_{\beta_1}\ d(p_{\beta_1}d(p_{\alpha_1},\ p_{ij})<w_{\beta_1}\ d(p_{\beta_1},\ p_{kl})$ (4)

By (2) and (3), $w_{\beta_1}\ d(p_{\beta_1}d(p_{\alpha_1},\ p_{kl})\ <w_{\alpha_1}d(p_{\alpha_1},\ p_{ij})$, and

by (1) and (4) $w_{\alpha_1}\ d(p_{\alpha_1},\ p_{ij})<w_{\beta_1}d(p_{\beta_1},\ p_{kl})$, which is impossible.

Now suppose $p_{ij}\in l(i,j)$. Obviously $\omega_id(p_i,p_{ij})=\max_{p_k\in P}\{\omega_kd(p_k,p_{ij})\}$, and for any other point x, either $d(p_i,p_{ij})<d(p_i,x)$ or $d(p_j,p_{ij})<d(p_j,x)$. Thus the lemma is proved.

Theorem.

Let V be the set of all Voronoi points in WFVD which is constructed for a given P. Suppose $\hat{p}$ is an internal division point which appears on the corresponding Voronoi line. Then the optimum point p^* is in $V\cup\{\hat{p}\}$.

Proof. We show that p^* lies on the WFVD as follows: Suppose the optimum point $p^*\notin$ WFVD. Then p^* must lie inside one of the WFVRs, say $p^*\in\hat{R}_i$. Let p' be the intersection point between the straight line $\overline{p_ip^*}$ and the boundary of $\hat{R}_i$ (p' always exists since $p_i\notin\hat{R}_i$ and the boundary of $\hat{R}_i$ consists of Voronoi lines, $\bigcup_j l(i,j)$), Then p' lies on a Voronoi line, say $l(i,k)$. Since $d(p_i,p')<d(p_i,p^*)$ and $\omega_id(p_i,p')\geq\ \omega_jd(p_j,p')$ for all $p_j\in P$, there exists a point p' whose cost $f^c(p')<f^c(p^*)$. This leads to a contradiction. We already proved in the previous lemma that if $\hat{p}$ exists, then $p^*=\hat{p}$. Thus we assume that $\hat{p}$ does not appear on the corresponding Voronoi line. Suppose the optimum point lies on the Voronoi line $l(i,j)$ but is not a Voronol point. Let v_{ijk} be a Voronoi point which is incident to $l(i,j)$ and is closer to the straight line segment $\overline{p_ip_j}$ than the other Voronoi point. Then it is clear that $f^c\ (p^*)=\omega_id(p_i,p^*)>f^c(v_{ijk})=\omega_id(p_i,v_{ijk})$, Thus p^* lies on one of the Voronoi points.

So far we know that the optimum location is determined by either two points or three points: If the optimum location is determined by two points, the optimum point is located at the internal division point between those two points. If it is determined by three points (say p_A,p_B and p_C), then the optimum point is located at the Voronoi point, v_{ABC}, which is inside the triangle $\Delta p_A p_B p_C$, of the WFVD with respect to those three points. Note that in the latter case, there is only one Voronoi point inside the triangle and this point is also a Voronoi point of the WFVD with respect to P. This leads to a straightforward algorithm to solve the Euclidean center problem.

Algorithm.

Given a set of n points $P=\{p_1, p_2, \ldots, p_n\}$ with ω_i's,

STEP 0: Pick any point $p_A \in P$ and Find $p_B \in P$ such that $\omega_B d(p_B, p_A) \geq \max_i \{\omega_i d(p_i, p_A)\}$ for all $p_i \in P$.

Find the internal division point, p_{AB}, between p_A and p_B.

STEP 1: $p^* \leftarrow p_{AB}$ and $f(p^*) \leftarrow \omega_A d(p_A, p^*)$

Find p_C such that $\omega_C d(p_C, p^*) = \max_i \{\omega_i d(p_i, p^*)\}$.

If $f(p^*) \geq \omega_C d(p_C, p^*)$, p^* is the optimum point, STOP

otherwise, draw WFVD (p_A, p_B) and find out which Voronoi Region includes p_C, say $\hat{R}_A$.

//If $p_C \in \hat{R}_B$, then p_A and $\hat{R}_A$ are interchanged with p_B and $\hat{R}_B$, respectively so that $\omega_A d(p_A, p_C) \geq \omega_B d(p_B, p_C)$//

If $p_{AC} \in \hat{R}_A$, then $p_B \leftarrow p_C$, GO TO STEP 1

otherwise GO TO STEP 2

// If $p_{AC} \in \hat{R}_A$, then either Voronoi point p_{ABC} in WFVD $(\{P_A, p_B, p_C\})$ is outside of $\Delta p_A p_B p_C$ or p_{ABC} does not appear in WFVD $(\{p_A, p_B, p_C\})$//

STEP 2: Draw WFVD (p_A, p_B, p_C) and Find Voronoi point p_{ABC} which is inside the triangle $\Delta p_A p_B p_C$. $p^* \leftarrow p_{ABC}, f(p^*) \leftarrow \omega_A d(p_A, p_{ABC})$

Find p_D such that $\omega_D d(p_D, p^*) = \max_i \{\omega_i d(p_i, p^*)\}$.

If $f(p^*) \geq w_D d(p_D, p^*)$, p^* is the optimum point, STOP

otherwise, find out which Voronoi Region includes p_D, say $\hat{R}_A$.

If i) $p_{AD} \in \hat{R}_A$, then $p_B \leftarrow p_D$, GO TO STEP 1

ii) $p_{AD} \in \hat{R}_B$, then $p_C \leftarrow p_D$, GO TO STEP 2

iii) $p_{AD} \in \hat{R}_C$, then $p_B \leftarrow p_D$, GO TO STEP 2.

Note that whenever new WFVD is obtained, we get $f(p^*)$ whose value is monotone increasing. Since there are only finitely many two-point and three-point sets, the process is finite.

Example: Consider the set of 7 points as shown in Figure 1

STEP 0: Suppose $p_1=(0, 0)$ is chosen as p_A. Then we have $p_B=p_5$ and $p_{AB}=(5.833, 1.667)$

STEP 1: $p^*=p_{AB}$, $f(p^*)=6.067$ and $p_C=p_2$ are obtained.

Since $f(p^*) < \omega_2 d(p_2, p^*) = 14.8$ and $p_C \notin \hat{R}_A$,

p_A is interchanged with p_B *i.e.* we have $p_A=p_5$ and $p_B=p_1$.

Since $p_{AC}=(5.57, 3.714) \in \hat{R}_A$, p_B is replaced by p_C and return to STEP 1.

STEP 1: $p^*=p_{AB}=p_{52}$, $f(p^*)=11.157$ and $p_C=p_3$ are obtained.

Since $f(p^*) < \omega_3 d(p_3, p_{52})$ and $p_{AC}=(5.875, 1.25) \in \hat{R}_B$, go to STEP 2.

STEP 2: $p^*=p_{ABC}=p_{523}=(5.327, 3.492)$ and $f(p^*)=11.206$ are obtained.

Note that in this step the Voronoi point p_{ABC} of WFVD $(\{p_A, p_B, p_C\})$ exists and is always inside of the triangle $\Delta p_A p_B p_C$

Since $f(p^*)=\max_i\{\omega_i d(p_i, p^*)\}$, we finally found the optimum point $p^*=(5.327,\ 3.492)$ with $f(p^*)=11.206$

4. EUCLIDEAN M-CENTER PROBLEM

It is natural to consider locating several new facilities instead of a single facility. The Euclidean m-center problem can be formulated as follows:

$$\text{minimize } f(X)=\max_{p_i\in P}\ \{\omega_i\cdot d(p_i, X)\}$$

where $P=\{p_1, p_2, \ldots, p_n\}$ is a given set of points in the plane representing existing n locations, ω_i is a given non-negative weight on point p_i representing interaction between p_i and one of the centers, $X=\{x_1, x_2, \ldots, x_m\}$ is a set of points in the plane to be determined as the site of new facilities, and $d(p_i, X)=\min_{1\leq j\leq m}\{d(p_i, x_j)\}$.

This problem is known to be NP-complete [7]. Therefore, we suggest one of the possible heuristic approaches. First, we will reduce Euclidean m-center problem to the minimum set covering problem, and then apply a heuristic algorithm for the set covering problem.

Let $X=\{x_1, x_2, \ldots, x_m\}$ be an optimum solution to m-center problem. Let us define $V(x_i)=\{p_j\in P\,|\,d(p_i, x_i)\ =d(p_j, X)\}$ and $f(x_i)=\max_{p_j\in V(x_i)}\{w_j d(p_i, x_i)\}$ Then, $f(X)=\max_{x_i}\{(x_i)\}$ and $P=\bigcup_i V(x_i)$. Note that one may regard each x_i as the center of $V(x_i)$. Since any "local center" for $P'\subset P$ is determined by either two points or three points in $P'(x_i=P'$ if $|P'|=1)$, the number of all the possible for candidates for center positions is finite: Let Y_1 be the set of all possible internal division points and Y_2 be the set of all Voronoi points, which lie inside the corresponding triangle, of the WFVD with respect to any three points in P. If we let Z be the set of all possible local centers, then $Z=P\cup Y_1\cup Y_2$ Let p_e be an "extreme" point which determines local center $z_i\in Z$. Let us define $r_i=\omega_e\ d(p_e, z_i)$ and $\Gamma z_i=\{p_j\in P\,|\,\omega_j\ d(p_j, z_i)\geq r_i\}$. Then r_i and Γz_i represent the radius of z_i and the set of points "covered" by z_i, respectively. Without loss of generality, assume that $Z=\{z_1, z_2, \ldots, z_t\}$ be ordered set such that the corresponding values, r_i's, are in an increasing. order. Note that an optimum value of $f(X)$ is one of the r_i's and in particular if $m=1, f(X)=r_t$. Therefore if we assume that an optimum solution $X=\{z_{i_1}, z_{i_2}, \ldots, z_{i_m}\}$ is an ordered set, X has the following properties:

(1) X covers P, thatis, $\Gamma X=P$.

(2) For any subset of Z, $Z'=\{z_1, z_2, \ldots, z_s\}$ with $r_s<r_{i_m}$,
there does not exist a cover of P with its cardinality less than or equal to m.

Algorithm.

STEP 0: Find all the local centers $z_1, z_2, \ldots, z_t$ with corresponding r_i's and Γz_i's. Define $Z_s=\{z_1, z_2, \ldots, z_s\}$ with appropriately assigned integer to $s<t$.

STEP1: Find minimum cover, X, of P for a given Z_s.
IF $|X|>m$, then increase index s by proper amount, GO TO STEP 1.
IF $|X|<m$, then decrease s by proper amount, GO TO STEP 1.
IF $|X|=m$, GO TO NEXT STEP.

STEP2: Let s' be the maximum index in X.
$s\leftarrow s'-1$, $X^*\leftarrow X$.

Find minimum cover X of P for a given z_s.

IF $|X|>m$, then X^* is an optimum solution, STOP

otherwise repeat STEP2.

Since the problem of finding a minimum cover is also known to be NP-complete, a heuristic algorithm for the set covering problem may be employed as a subproblem. Note that the time complexity of the proposed algorithm depends heavily on the cardinality of Z which is obviously in the order of n^3. As an example, the number of all the possible local centers for the previous example ($n=7$) is 40 as detailed in Table1. Thus, it may be important to determine an initial index s in the proposed algorithm. There are two possible ways: it may be determined by (1) a proper value of r_s and (2) a proper index assumed as a function of n(say, $s=(n-m+1)n, m\geq 2$).

Table 1

z_i	p_e	Location of z_i	r_i	Γz_i	z_i	p_e	Location of z_i	r_i	Γz_i
1	1	(0, 0)	0	1	21	3, 7	(6, 0)	6.0	1, 3, 7
2	2	(2, 8)	0	2	22	1, 2, 4	(3.069, 5.184)	6.024	1, 2, 4, 6
3	3	(4, 0)	0	3	23	1, 5	5.833, 1.667)	6.097	1, 5
4	4	(4, 4)	0	4	24	1, 5, 6	(5.777, 2.008)	6.116	1, 5, 6
5	5	(7, 2)	0	5	25	1, 5, 7	(6.108, 1.135)	6.213	1, 5, 7
6	6	(7, 8)	0	6	26	3, 6	(4.75, 2)	6,408	1, 3, 6
7	7	(7, 0)	0	7	27	1, 6, 7	(6.222, 1.618)	6.429	1, 5, 6, 7
8	1, 3	(3, 0)	3.0	1, 3	28	3.5	(5.875, 1.25)	6.760	1, 3, 5, 7
9	2, 6	(3.667, 8)	3.333	2, 6	29	3, 5, 6	(5.822, 1.322)	6.771	1, 3, 5, 6
10	4, 6	(4.6, 4.8)	4.0	4, 6	30	3, 6, 7	(5.858, 1.298)	6.799	1, 3, 5, 6, 7
11	5, 7	(7.571, 1, 429)	4.04	5, 7	31	3.4	(4, 2.286)	6.857	1, 3, 4, 6
12	1, 4	(3.3, 3.2)	4.525	1, 4	32	4, 5	(5.667, 2.889)	8.012	1, 4, 5, 6
13	5, 6	(7, 3)	5.0	5, 6	33	3, 4, 5	(5.380, 2.426)	8.374	1, 3, 4, 5, 6
14	1, 6	(3.5, 4)	5.315	1, 4, 6	34	4, 7	(5.667, 2.667)	8.537	1, 4, 5, 6, 7
15	1, 2	(1.33, 5.33)	5.497	1, 2	35	3, 4, 7	(5.478, 2.447)	8.577	1, 3, 4, 5, 6, 7
16	6, 7	(8.33, 2.66)	5.497	6, 7	36	2, 3	(3.2, 3.2)	9.895	1, 2, 3, 4, 6
17	5, 6, 7	(7.954, 2.557)	5.526	5, 6, 7	37	2, 7	(5.5, 4)	10.630	1, 2, 4, 6, 7
18	1, 2, 6	(2.134, 5.195)	5.616	1, 2, 6	38	2, 3, 7	(4.895, 3.470)	10.751	1, 2, 3, 4, 6, 7
19	2, 4	(3.33, 5.33)	5.963	2, 4, 6	39	2, 5	(5.57, 3.714)	11.157	1, 2, 4, 5, 6, 7
20	1, 7	(6.0)	6.0	1, 3, 7	40	2, 3, 5	(5.327, 3.492)	11.206	1, 2, 3, 4, 5, 6, 7

1-Center	z_{40}	11.206
2-Center	z_{28} z_{19}	6.760
3-Center	z_{19} z_{11} z_8	5.963
4-Center	z_{11} z_{10} z_8 z_2	4.04
5-Center	z_9 z_8 z_7 z_5 z_4	3.33
6-Center	$z_8 z_7$ z_6 z_5 z_4 z_2	3.0

V. FURTHER REMARKS

As a final remark, there is another extension of the Euclidean center problem, calledthe Euclidean minimax multifacility location problem, This problem can be formulated as follows:
Minimize max $\{\omega_{ji}d(p_i, x_j) j=1, \ldots, m,\ i=1, \ldots, n:\ u_{jk}d(x_j, x_k)\ 1\leq j<k\leq m\}$

where $P=\{p_1, p_2, \ldots, p_n\}$ is a given set of points in the plane representing the locations of existing facilities, $X=\{x_1, x_2, \ldots, x_n\}$ is a set of points in the plane to be determined as the site of new facilities, ω_{ji} is a given non-negative weight representing interaction between the new facility at x_j and the existing facility at p_i, and u_{jk} is a given non-negative weight representing interaction between new facilities (x_j and x_k).

Love et al [8] and Elzinga et al [9] attacked this problem using non-linear programming methods. From their computational results in [9] it can be seen that if the number of primal constraints, $mn+m(m-1)/2$, is less than 200, the problem can probably be solved in a few minutes of computer time.

Note that if $m=1$, this problem becomes the Euclidean center problem which is solved in section 3. Also if $u_{jk}=0$ for all j and k, this problem can be solved by simply repeating our algorithm in the previous section m times. Thus there may be some future in solving the general problem by considering and adjusting the solution which is obtained assuming that $u_{jk}=0$ for all j and k.

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