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A Study on the Spherical Indicatrix of a Space Curve in E^3

By Sang Kie Chung
Seoul High School

Abstract: Many interesting properties of a space curve C in E^3 may be investigated by means of the concept of spherical indicatrix of tangent, principal normal, or binormal to C. The purpose of the present paper is to derive the representations of the Frenet frame field, curvature, and torsion of spherical indicatrix to C in terms of the quantities associated with C. Furthermore, several interesting properties of spherical indicatrix are found in the present paper.

I. Introduction.

Many interesting properties of a space curve C in a 3-dimensional Euclidean space E^3 may be investigated by means of the concept of spherical indicatrix C_1 of tangent, C_2 of principal normal, and C_3 of binormal to C, respectively. In the present paper, we derive the representations of the Frenet frame field, curvature, and torsion of spherical indicatrix C_i, $(i=1,2,3)$, to C in terms of the quantities associated with C, first of all. In the second, the geometrical interpretations of these representations are remarked. Finally, some interesting properties of spherical indicatrix are found in the present paper.

II. The Indicatrix of a Space Curve in E^3.

This section is concerned with an introduction to the essential concepts and results of the theory of space curves in a 3-dimensional Euclidean space E^3, which are needed in our considerations in the present paper.

Let

(2.1) $$\boldsymbol{x}=\boldsymbol{x}(s)$$

be a natural representation of a regular curve C in E^3. The relations

(2.2) $$\boldsymbol{t}'=\kappa\boldsymbol{n},\quad \boldsymbol{n}'=-\kappa\boldsymbol{t}+\tau\boldsymbol{b},\quad \boldsymbol{b}'=-\tau\boldsymbol{n}$$

are known as the *Serret-Frenet formulae*, where t, n, b, κ, and τ represent the tangent, the principal normal, the binormal, the curvature, and the torsion of C, respectively. Here the prime signifies differentiation with respect to s. These relations undelie many investigations in the theory of curves and surfaces in E^3.

The curvature and the torsion of C may be given by the following formulae:

(2.3) $$\kappa^2=\boldsymbol{t}'\cdot\boldsymbol{t}'=\boldsymbol{x}''\cdot\boldsymbol{x}''$$

$$\tau=\frac{(\boldsymbol{t}\boldsymbol{t}'\boldsymbol{t}'')}{\boldsymbol{t}'\cdot\boldsymbol{t}'}=\frac{(\boldsymbol{x}'\boldsymbol{x}''\boldsymbol{x}''')}{\boldsymbol{x}''\cdot\boldsymbol{x}''} \tag{2.4}$$

The locus $\begin{cases} C_1 \\ C_2 \\ C_3 \end{cases}$ of a point whose position vector is $\begin{cases} \text{the tangent } \boldsymbol{t} \\ \text{the principal normal } \boldsymbol{n} \\ \text{the binormal } \boldsymbol{b} \end{cases}$ of a curve C is called the *spherical indicatrix* of $\begin{cases} \text{the tangent} \\ \text{the princiapal normal} \\ \text{the binormal} \end{cases}$ to C. Obviously the spherical indicatrix is a spherical curve on a unit sphere about the origin. Its equation may be given by

$$C_1:\ \boldsymbol{x}_1=\boldsymbol{x}_1(s_1)=\boldsymbol{t}, \tag{2.5}$$

$$C_2:\ \boldsymbol{x}_2=\boldsymbol{x}_2(s_2)=\boldsymbol{n}, \tag{2.6}$$

$$C_3:\ \boldsymbol{x}_3=\boldsymbol{x}_3(s_3)=\boldsymbol{b}, \tag{2.7}$$

respectively.

III. Some Results.

Many properties of a space curve C in E^3 are conveniently investigated by means of the concept of spherical indicatrix. In the present section, some elementary properties of the spherical indicatrix and several useful relations are obtained.

Theorem (3.1). *The curvature and the torsion of C satisfy the following relations:*

$$\left|\frac{ds_1}{ds}\right|=|\kappa|, \tag{3.1)a}$$

$$\left|\frac{ds_2}{ds}\right|=\sqrt{\kappa^2+\tau^2}\ \stackrel{def}{=}\ \sigma, \tag{3.1)b}$$

$$\left|\frac{ds_3}{ds}\right|=|\tau|. \tag{3.1)c}$$

Proof. In virtue of (2.5), (2.6), (2.7), and (2.2), we have

$$\boldsymbol{t}_1=\frac{d\boldsymbol{x}_1}{ds_1}=\kappa\boldsymbol{n}\frac{ds}{ds_1}, \tag{3.2)a}$$

$$\boldsymbol{t}_2=\frac{d\boldsymbol{x}_2}{ds_2}=(-\kappa\boldsymbol{t}+\tau\boldsymbol{b})\frac{ds}{ds_2}, \tag{3.2)b}$$

$$\boldsymbol{t}_3=\frac{d\boldsymbol{x}_3}{ds_3}=-\tau\boldsymbol{n}\frac{ds}{ds_3}. \tag{3.2)c}$$

The relations (3.1) are direct consequences of (3.2).

Agreement (3.2). Throughout this paper we shall assume, without the loss of generality, that the sign of (3.1) be taken in such a way that

$$\frac{ds_1}{ds}=\kappa,\ \frac{ds_2}{ds}=\sigma,\ \frac{ds_3}{ds}=\tau. \tag{3.3}$$

Theorem (3.3). *The tangent* $\boldsymbol{t}_1$, *the principal normal* $\boldsymbol{n}_1$, *the binormal* $\boldsymbol{b}_1$, *the curvature* κ_1, *and the torsion* τ_1 *of* C_1 *may be respectively given by*

$$\boldsymbol{t}_1=\boldsymbol{n}, \tag{3.4)a}$$

$$\boldsymbol{n}_1=\pm\frac{-\kappa\boldsymbol{t}+\kappa\boldsymbol{b}}{\sigma}, \tag{3.4)b}$$

$$\boldsymbol{b}_1=\pm\frac{\kappa\boldsymbol{b}+\tau\boldsymbol{t}}{\sigma}, \tag{3.4)c}$$

(3.5)a $$(\kappa_1)^2=\left(\frac{\sigma}{\kappa}\right)^2,$$

(3.5)b $$\tau_1=\frac{\kappa\tau'-\kappa'\tau}{\kappa\sigma^2.},$$

Proof. The first relation (3.4)a is a direct consequence of (3.2)a and (3.3). Differentiating both sides of (3.4)a with respect to s_1 and making use of (2.2) and (3.3), we have

$$\kappa_1\boldsymbol{n}_1=\frac{d\boldsymbol{t}_1}{ds_1}=(-\kappa\boldsymbol{t}+\tau\boldsymbol{b})\frac{ds}{ds_1}=\frac{-\kappa\boldsymbol{t}+\tau\boldsymbol{b}}{\kappa},$$

which gives the represention (3.5)a and (3.4)b in virtue of (2.3). The vector product of (3.4)a and (3.4)b now gives (3.4)c. Finally, differentiating both sides of (3.4)c with respect to s_1 and making use of (2.2), (3.3), and (3.4)b, we have

$$-\tau_1\boldsymbol{n}_1=\frac{d\boldsymbol{b}_1}{ds_1}=\pm\frac{\kappa'\tau-\kappa\tau'}{\sigma^3}(-\kappa\boldsymbol{t}+\tau\boldsymbol{b})\frac{ds}{ds_1}$$
$$=\frac{\kappa'\tau-\kappa\tau'}{\sigma^2\kappa}\boldsymbol{n}_1,$$

from which the representation (3.5)b follows.

The following two theorems will be proved simultaneously.

Theorem (3.4). *The tangent $\boldsymbol{t}_2$, the principal normal $\boldsymbol{n}_2$, the binormal $\boldsymbol{b}_2$, the curvature κ_2, and the torsion τ_2 of C_2 may be respectively given by*

(3.6)a $$\boldsymbol{t}_2=\frac{-\kappa\boldsymbol{t}+\tau\boldsymbol{b}}{\sigma},$$

(3.6)b $$\kappa_2\boldsymbol{n}_2=-\boldsymbol{n}-\frac{\kappa'\tau-\kappa\tau'}{\sigma^4}(\tau\boldsymbol{t}+\kappa\boldsymbol{b}),$$

(3.6)c $$\kappa_2\boldsymbol{b}_2=\frac{\kappa\boldsymbol{b}+\tau\boldsymbol{t}}{\sigma}-\frac{\kappa'\tau-\kappa\tau'}{\sigma^3}\boldsymbol{n},$$

(3.7)a $$(\kappa_2)^2=1+\left(\frac{\kappa'\tau-\kappa\tau'}{\sigma^3}\right)^2,$$

(3.7)b $$\tau_2=\frac{-\rho'-\rho(\kappa\kappa'+\tau\tau')/\sigma^2}{(\kappa_2)^2},$$

where

(3.8) $$\rho\overset{def}{=\!=}\frac{\kappa'\tau-\kappa\tau'}{\sigma^4}.$$

Theorem (3.5). *The tangent $\boldsymbol{t}_3$, the principal normal $\boldsymbol{n}_3$, the binormal $\boldsymbol{b}_3$, the curvature κ_3, and the tortion τ_3 of C_3 may be respectively given by*

(3.9)a $$\boldsymbol{t}_3=-\boldsymbol{n},$$

(3.9)b $$\boldsymbol{n}_3=\pm\frac{\kappa\boldsymbol{t}-\tau\boldsymbol{b}}{\sigma},$$

(3.9)c $$\boldsymbol{b}_3=\pm\frac{\kappa\boldsymbol{b}+\tau\boldsymbol{t}}{\sigma},$$

(3.10)a $$(\kappa_3)^2=\left(\frac{\sigma}{\tau}\right)^2,$$

(3.10)b $$\tau_3=\frac{\kappa'\tau-\kappa\tau'}{\tau\sigma^2}.$$

Proof. The proofs of the above two theorems are almost similar to that of Theorem (3.3) except the relation (3.7)b, the proof of which follows from (2.4) and the following result obtained from (3.6)a, b:

$$\begin{aligned} & \boldsymbol{x}_2' \cdot (\boldsymbol{x}_2'' \times \boldsymbol{x}_2''') \\ & = \frac{1}{\sigma^2}(-\kappa \boldsymbol{t} + \tau \boldsymbol{b}) \cdot [(\rho\tau \boldsymbol{t} + \boldsymbol{n} + \rho\kappa \boldsymbol{b}) \times \{(-\kappa + (\rho\tau)')\boldsymbol{t} + (\tau + (\rho\kappa)')\boldsymbol{b}\}] \\ & = -\rho' - \frac{\rho(\kappa\kappa' + \tau\tau')}{\sigma^2}. \end{aligned}$$

Remark (3.6). In virtue of the preceding three theorems, we note the following remarks:

(a) The tangents to C_1 and C_3 are both parallel to the principal normal to C at corresponding points.

(b) The principal normal to C_1, the tangent to C_2, and the principal normal to C_3 are parallel each other at corresponding points, and they all lie on the rectifying plane of C at corresponding points.

(c) The binormals to C_1 and C_3 are parallel each other and lie on the rectifying plane of C at corresponding points.

The following two theorems will be proved simultaneously.

Theorem (3.7). *The spherical indicatrix C_1 is a circle if and only if the curve C is a helix.*

Theorem (3.8). *The spherical indicatrix C_3 is a circle if and only if the curve C is a helix.*

Proof Making use of (3.5)a and (3.5)b, Theorem (3.7) may be proved by the following argument:

C_1 is a circle iff $(\kappa_1)^2 = 1 + \left(\frac{\tau}{k}\right)^2 =$ constant, $\tau_1 = \frac{\kappa\tau' - \kappa'\tau}{\kappa\sigma^2} = 0$ iff $\frac{\tau}{\kappa} =$ constant iff C is a helix.

The proof of Theorem (3.8) is similar.

Remark (3.9). We also note that, in virtue of (3.7)a and (3.7)b, the spherical indicatrix C_2 is a circle with radius 1 if the corresponding curve C is a circular helix whose intrinsic equations are

$$\kappa = \text{constant}, \quad \tau = \text{constant}.$$

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