THE JACOBSON DENSITY THEOREM AND SOME APPLICATIONS

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1. The density theorem

Consider the ring $Q = \text{Hom}(M, M)$, where $M$ is an abelian group. Let $R$ and $S$ be rings contained in $Q$, such that $rM$ is an irreducible $R$-module and such that $S = \{ a \in Q | ar = ra \text{ for all } r \in R \}$. Then $S$ is a division ring and if $M_S$ is a finite dimensional vector space, $R$ is isomorphic to a ring of matrices over $S$.

This result can be generalized as in the following two theorems. Furthermore it will be clear that if $R$ is a primitive ring and if $rM$ is a faithful simple $R$-module, then $R$ and $rM$ satisfy the conditions of theorems 1 and 2, so that the usual Jacobson density theorem is a consequence of theorem 2.

**THEOREM 1.** Let $R$ be a ring, $rM$ a faithful $R$-module, $S = \text{Hom}_R(rM, rM)$ and $M_S$ the corresponding $S$-module. Suppose that if $N$ is a proper submodule of $rM$, then there is a nonzero element $f$ of $S$ such that $Nf = 0$. Also, suppose that if $a \in M$, $Ra = 0$, then $a = 0$.

If $V$ is a finite dimensional free $S$-submodule of $M$, if $a \in M$ and if $A(V) = \{ x \in R | xV = 0 \}$, then $A(V)a = 0$ implies $a \in V$. If a basis of $V$ together with $a$ forms an $S$-linearly independent set, then $A(V)a = M$.

**THEOREM 2.** Let $R, S, rM$ and $M_S$ be as in theorem 1 and suppose that the hypotheses of theorem 1 hold. Let $M_S$ be a free $S$-module. Then there is a monomorphism $\xi$ from $R$ to $R^* = \text{Hom}_S(M_S, M_S)$ such that $\xi(R)$ is dense in $R^*$.

**Proof of theorem 2.** Identify $R$ as a subring of $R^*$. Let $U = \{ u_1, \ldots, u_m \}$ be a linearly independent subset of $M$ over $S$ and let $\{ v_1, \ldots, v_m \}$ be a subset of $M$. If $W_j$ is the $s$-span of $\{ u_1, \ldots, u_j, \ldots, u_m \}$, then $A(W_j)u_j = M$, and hence there is an element $r_j$ of $A(W_j)$ such that $r_ju_j = v_j$ and $r_ju_m = 0$ if $i + j$. If $r = \sum ir_i$, then $ru_j = \sum ir_iu_j = r_ju_j = v_j$ for all $j$. Hence $R$ is dense in $R^*$.

**Proof of theorem 1.** Suppose $V = \{ 0 \}$, then $A(V) = R$, so that $Ra = 0$ implies $a = 0$. Also, if $\{ a \}$ is linearly independent over $S$, then $Ra = M$, since

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otherwise \((Ra)f=0\), whence \((a)f=af=0\), contradicting the fact that \{a\} is linearly independent over \(S\). Now, if \(V=W+uS\), with \(u\) \(S\)-linearly independent of \((\text{a basis of})\) \(W\), and if the conclusions of the theorem hold for \(W\), then \(A(W)u=M\). Suppose this is so and that \(A(V)a=0\). If \(x\in A(W)\) and \(xu=0\), then \(x\in A(V)\), so that also \(xa=0\). Thus, there is an \(R\)-homomorphism \(\theta\) from \(A(W)u\) to \(A(W)a\) given by \(\theta:xu\to xa\) for all \(x\in A(W)\).

Since \(A(W)\cdot(u\theta-a)=0\), \(u\theta-a=-w\in W\), so that \(w+u\theta=a\in V\). The first conclusion follows.

Next, suppose that a basis of \(V\) together with \(a\) forms an \(S\)-linearly independent set. Also suppose that \(A(V)a\not\approx M\). Hence there is a nonzero element \(f\) of \(S\) such that \((A(V)a)f=A(V)(af)=0\). Since \(f\not=0\) and since \{a\} is \(S\)-linearly independent \(af\not=0\).

Consider the map \(\theta\) from \(A(W)u\) to \(A(W)(af)\) given by \(\theta:xu\to x(af)\). Note that \(A(W)u=M\) by the induction hypothesis. Let \(x\in A(W)\); then if \(xu=0\), \(x\in A(V)\) and \(x(af)=0\). Hence \(\theta\) is a well-defined \(R\)-endomorphism of \(M\), i.e., \(\theta\in S\). Moreover, \(A(W)(u-af)=0\), so that \(u\theta-af\in W\). But then \(af\not=0\) and \(af\in V\) contradicting the hypothesis that \(a\) is \(S\)-linearly independent from \(V\). Hence \(A(V)a=M\) and the theorem follows.

2. Applications

We use theorems 1 and 2 to discuss some notions and theorems extending the usual notions of primitive ring and irreducible module as well as some of the theory relating them. Let \(rM\) be a faithful \(R\)-module and let \(S=\text{Hom}_R(M,M)\). If for any \(m\in M\), \(Rm=0\) implies \(m=0\), and if for any proper submodule \(N\) of \(rM\), \(Nf=0\) for some nonzero \(f\in S\), while also \(M_S\) is a free \(S\)-module, then \(R\) is \(R\)-almost primitive ring and \(rM\) is a characteristic \(R\)-module.

From theorems 1 and 2 it follows via the usual argument that if \(R\) is \(R\)-almost primitive ring, then \(R\) has an epimorphic image of the ring of all finite-by-finite matrices over \(S\).

For the converse we have the following.

**Theorem 3.** Let \(S\) be a ring with identity such that for each proper left ideal \(I\) there is a nonzero element \(a\) of \(S\) such that \(Ia=0\). Then \(R\), the complete ring of finite-by-finite matrices over \(S\), is a \(R\)-almost primitive ring.

**Proof.** Let \(E_{ij}\) be the matrix with \((i,j)\) entry 1, and all other entries 0. Let \(M=RE_{11}\). Then, \(M=E_{11}S\oplus E_{21}S\oplus \cdots \oplus E_{n1}S\oplus \cdots\), whence \(M\) is a free right \(S\)-module as well as a left \(R\)-module. We claim that \(\text{Hom}_R(M,M)\)=\(S\).

Thus, let \(g\in \text{Hom}_R(M,M)\) and let \((E_{ii})g=\sum x_i E_{ii}, \ x_i\in S\). Therefore,
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\[(E_{11}) g = (E_{11} \cdot E_{11}) g = E_{11} (E_{11}) g = E_{11} \cdot \sum E_{ij} x_i = E_{11} x_1.\] Now if \(m \in M, m = y E_{11}\) for some \(y \in R\), then \((y E_{11}) g = y (E_{11}) g = y E_{11} x_1\), whence \(mg = mx\), for all \(m \in M\), i.e., \(g \in S\).

Now, let \(p\) be a proper submodule of \(pM\). We claim that \(Nf = 0\) for some nonzero element \(f\) of \(S\). Let \(m = \sum E_{ij} x_i, x_i \in S\), be an element of the proper submodule \(pN\) of \(pM\). Then, \(E_{ij} m = E_{ij} x_i\) and \(N = E_{11} \oplus E_{21} \oplus \cdots\) for some left ideal \(I\) of \(S\). Since \(I\) is proper, \(fa = 0\) for some nonzero element \(a\) of \(S\). Hence, \(Na = 0\), and it suffices to take \(f = a\). Clearly, \(Rm = 0\) implies \(m = 0\). Thus, \(R\) is an almost primitive ring with characteristic module \(pM\).

If \(S\) is a (right) Steinitz ring, i.e., a local ring with radical \(J\) such that for each sequence \(\{x_i\}\) of elements of \(J\) there is an \(n\) such that \(x_n x_{n-1} \cdots x_1 = 0\), then \(Ja = 0\) for some nonzero element \(a\) of \(S\). Hence, \(Ja = 0\) if \(I\) is a proper left ideal. Thus theorem 3 has the following:

**Corollary** The complete ring of finite-by-finite matrices over a (right) Steinitz ring \(S\) is a (left) almost primitive ring.

Collecting results we obtain:

**Theorem 4.** If \(S\) is a ring with identity such that for each proper left ideal \(I\) there is a nonzero element \(a\) of \(S\) such that \(Ia = 0\), then the ring \(R\) of all finite-by-finite matrices over \(S\) is (left) almost primitive.

Conversely, if \(R\) is a (left) almost primitive ring, then \(R\) is epimorphic to a ring for which the ring of all finite-by-finite matrices over a ring \(S\) with identity, such that for each left ideal \(I\) of \(S\) there is a nonzero element \(a\) of \(S\) such that \(Ia = 0\), is a dense set.

**Proof:** Let \(R\) be (left) almost primitive and let \(S = \Hom_R(M, M)\), where \(pM\) is a characteristic \(R\)-module. Let \(\{m_1, m_2, \ldots\}\) be an \(S\)-basis of \(M\). Let \(I\) be a proper left ideal of \(S\). Then \(N = \sum m_i I\) is a proper \(R\)-submodule of \(M\). Hence \(Nf = 0\) for some nonzero \(f \in S\). Thus \(If = 0\) and \(f = a\) will do.

3. Artinian almost primitive rings

If \(R\) is Artinian and almost primitive, with characteristic module \(pM, S = \Hom_R(M, M)\), we let \(J(R)\) be the Jacobson radical of \(R\). Now \(R\) is isomorphic to the complete ring of \(n \times n\) matrices over \(S\) for some finite number \(n\), say \(R = S_n\). Thus \(J = J(R) = JS_n\), whence \(J = I_n\), the Jacobson radical of \(S\), \(J(S)\), being the ideal \(I\). Thus, \(R/J = (S/I)_n\) is semisimple and Artinian, so that it is a direct sum of finitely many ideals which are simple rings, Say \((S/I)_n = S_1 \oplus \cdots \oplus S_r\). Then \(S_i = (I_i)_n\) for some ideal \(I_i\) of \(S/I\), whence \((S/I)_n = (I_1)_n \oplus \cdots \oplus (I_r)_n\). Thus, \(S/I = I_1 \oplus \cdots \oplus I_r\) where each ideal \(I_i\)
is a division ring. It follows that:

**THEOREM 5.** If $R$ is Artinian and almost (left) primitive then $R=S_n$, the complete ring of $n \times n$ matrices, for some $n$, over $S=\text{Hom}_R(M, M)$. Furthermore if $I$ is the Jacobson radical of $S$, then $S/I=I_1 \oplus \cdots \oplus I_r$, where each ideal $I_r$ is a division ring. Also, $Ia=0$ for some nonzero element $a$ of $S$.

4. Artinian almost semisimple rings

Given a ring $R$, a two-sided ideal is a (left) almost primitive ideal if $R/I$ is a left almost primitive ring. We let $AJ(R)$ denote the intersection of all (left) almost primitive ideals. We shall say $R$ is (left) almost semisimple if $AJ(R)=0$.

Suppose $R$ is Artinian. Then $AJ(R)=I_1 \cap \cdots \cap I_n$ for some finite set of almost primitive ideals. If $R$ is Artinian (left) almost semisimple, then $AJ(R)=I_1 \cap \cdots \cap I_n=0$ and there is an injection $R \rightarrow R/I_1 \oplus \cdots \oplus R/I_n$, where the rings $R/I_i$ are themselves Artinian (left) almost primitive rings, i.e., rings as described in theorem 5. Thus, $R$ is a finite subdirect sum of (left) almost primitive rings. Under a variety of conditions this subdirect sum will be a direct sum, i.e., we have a Chinese Remainder Theorem to produce an isomorphism between $R$ and $R/I_1 \oplus \cdots \oplus R/I_n$.

If $R$ is a subdirect sum of $R_1 \oplus \cdots \oplus R_n$, where each $R_i$ is an Artinian (left) almost primitive ring, then $R$ is (left) almost semisimple. Indeed, $R_1=R/I_1 \cap R$, where $I_1=R_2 \oplus \cdots \oplus R_n$ is a (left) almost primitive ideal, so that $I_1 \cap R$ is a (left) almost primitive ideal of $R$. Hence $AJ(R)=0$ and $R$ is (left) almost semisimple. Since $R \cap R_i=R_i$, it follows that $R$ is Artinian. Thus, we have the following:

**THEOREM 6.** $R$ is Artinian and (left) almost semisimple if and only if $R$ is a subdirect sum of a finite number of Artinian (left) almost primitive rings.

**References**