ON CONNECTIONS WITH TORSIONS AND THOSE CURVATURE TENSORS IN RIEMANNIAN MANIFOLDS II*

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I. A characterization of a Kaehlerian manifold by a curvature tensor formed with an $F$-connection with torsion.

§ 1. Preliminaries.

We consider a Kaehlerian manifold $M$ of real $2n$ dimensions ($n \geq 2$) covered by a coordinate neighborhoods $\{U; x^h\}$ and denote by $g_{ji}$ and $F^h_{ij}$ components of the Hermitian metric tensor and those of the Kaehlerian structure tensor of $M$ respectively, where, here and in the sequel, the indices $h$, $i$, $j$, $\cdots$ run over the range $\{1, 2, \cdots, 2n\}$.

Let $'D$ be an affine connection with torsion in a Kaehlerian manifold $M$. We denote by $'\Gamma^k_{ij}$ the components of the connection $'D$ and by $'D_j$ the operator of covariant differentiation with respect to $'\Gamma^k_{ij}$.

If the affine connection $'D$ satisfies

\begin{align*}
(1.1) & \quad 'D_k g_{ji} = -2p_k g_{ji}, \\
(1.2) & \quad 'D_k F^h_{ji} = -2p_k F^h_{ji} \quad \text{(or $'D_k F^h_{ij} = 0$)}, \\
(1.3) & \quad '\Gamma^k_{ji} - '\Gamma^k_{ij} = -2F^h_{ji} q^h,
\end{align*}

where $F^h_{ji} = g^h_{ik} F^k_{ij}$, for a certain non zero covector field $p_k$ and a vector field $q^h$, then $'D$ is called a complex Weyl-Hlavaty connection [4].

Solving (1.1) and (1.3) with respect to $'\Gamma^k_{ij}$, it is easily obtained that

\begin{equation}
'\Gamma^k_{ji} = \{^h_{ji}\} + \partial^h_j p_i + \partial^h_i p_j - g_{ji} p^h + F^h_{ji} q_i + F^h_{ij} q_j - F^h_{ji} q^h,
\end{equation}

where $p^h = p_k g^{kh}$, $q^h = q_k g^{kh}$ and $\{^h_{ji}\}$ are the Christoffel symbols of $M$.

Computing $'D_k F^h_{ji}$ and taking account of (1.4), we obtain

\begin{equation}
(n-1) (p_i F^h_{ji} + q_i) = 0,
\end{equation}

from which

\begin{equation}
q_i = -p_i F^h_{ij}, \quad p_i = q_i F^h_{ij}.
\end{equation}

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In a previous paper \[1\], we considered an affine connection \(D\) whose components \(\Gamma_{ji}^h\) are related to those of complex Weyl-Hlavaty connection \(\Gamma'\) by
\[
\Gamma_{ji}^h = \Gamma'_{ji}^h - p_j \partial_i^h,
\]
that is,
\[
\Gamma_{ji}^h = \{^h_i\} + \partial_j p_i - g_{ji} p^h + F_j^h q_i + F_i^h q_j - F_{ji} q^h.
\]
in a Kaehlerian manifold \(M\).

(Such a connection \(D\) is called "a semi-symmetric metric \(F\)-connection" in \[5\].)

Denoting by \(R_{kji}^h\) the curvature tensor of \(\Gamma_{ji}^h\), we obtain
\[
R_{kji}^h = K_{kji}^h - \delta_k^h p_{ji} + \partial_j p_k + F_{kji} q^h - F_{ki} q_{ji} + \cdots,
\]
where \(K_{kji}^h\) is the Riemannian curvature tensor of \(M\),
\[
\lambda \text{ being defined by } \lambda = p_i p^i = q_i q^i.
\]

\[
\alpha_{ji} = -(F_{ji} - F_{ij}) + 2 (p_{ji} - q_{ji})
\]
and \(p_i^h = p_i g^{ih}\), \(q_i^h = q_i g^{ih}\), \(\beta_{ij}^h = \beta_{ji} g^{ih}\).

The following relations are easily checked.
\[
\partial_j p_i = q_{ji} F_i^h, \quad q_{ji} = - p_{ji} F_i^h,
\]
\[
\alpha_{ji} = -(q_{ji} - q_{ij} - \lambda F_{ji}).
\]

§ 2. A certain \(F\)-connection which is closely related to the complex Weyl-Hlavaty connection.

In this section, we use the fact that if
\[
\Gamma_{ji}^h = \{^h_i\} + T_{ji}^h,
\]
\(T_{ji}^h\) being a tensor field of type (1, 2), then the curvature tensor \(R_{kji}^h\) formed with \(\Gamma_{ji}^h\) is given by
\[
R_{kji}^h = K_{kji}^h + V_k T_{ji}^h - V_j T_{ki}^h + T_{ki}^h T_{ji}^h - T_{ji}^h T_{ki}^h.
\]

In this place, we want to seek out an \(F\)-connection \({^*D}\) in a Kaehlerian manifold \(M\) such that if the curvature tensor formed with the components of
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* D vanishes then $M$ is of constant holomorphic sectional curvature.

For this aim, it is desirable that the tensor $p_{ji}$ in (1.8) is symmetric. By this reason, we firstly put

\[(2.2) \quad \Gamma^{*}_{ji} = \Gamma^{h}_{ji} - p_{ji} F^{h}_{ij},\]

where $\Gamma^{h}_{ji}$ are the components of a metric connection $D$ defined by (1.7).

Denoting by $\Gamma^{(1)}_{ji}$, the operator of a metric connection with respect to $\Gamma^{h}_{ji}$, we find that $\Gamma^{(1)}_{ji}$ is an $F$-connection, that is, $\Gamma^{(1)}_{kl} F_{ij} = 0$ and the curvature tensor $\Gamma^{(1)}_{kj} F_{ij}$ is of the form:

\[(2.3) \quad \Gamma^{(1)}_{kj} F_{ij} = R^{(1)}_{kj} F_{ij} - (p_{kj} - p_{jk}) F^{h}_{ij},\]

where $R_{kj} F_{ij}$ is the curvature tensor of $\Gamma^{h}_{ji}$ defined by (1.7).

Secondly, it is desirable for our aim that the expecting $F$-connection annihilates the terms

$- F^{h}_{k} g_{ji} + F^{h}_{k} g_{ki}$ in (1.8). By this reason, we put

\[(2.4) \quad \Gamma^{(2)}_{ji} = \Gamma^{(1)}_{ji} - p^{h}_{ji} q_{i}.\]

Then the curvature tensor $\Gamma^{(2)}_{kj} F_{ij}$ of $\Gamma^{(2)}_{ji}$ is of the form

\[(2.5) \quad \Gamma^{(2)}_{kj} F_{ij} = (\Gamma^{(1)}_{kj} F_{ij} - F^{h}_{ij} (1.7) D_{g_{ij}} + F^{h}_{ij} (1.7) D_{g_{ij}})
- q_{i} (\Gamma^{(1)}_{kj} F_{ij} - (1.7) D_{g_{ij}} F_{ij})
- p^{h}_{q_{i}} F_{ij} - p^{h}_{q_{i}} F_{ij}.\]

From (1.6) and (2.2), we easily obtain

\[(2.6) \quad (1.7) D_{g_{ij}} - q_{i} (\Gamma^{(1)}_{kj} F_{ij} - (1.7) D_{g_{ij}} F_{ij})
- p^{h}_{q_{i}} F_{ij} - p^{h}_{q_{i}} F_{ij}.\]

Finally, it is also desirable that the expecting $F$-connection annihilates

the terms $- \delta h_{k} p_{ji} + \delta h_{j} p_{ki}$ in (1.8), so we put

\[(2.8) \quad \Gamma^{*}_{ji} = (2.2) \Gamma^{(1)}_{ji} - \delta h_{j} p_{i}.\]

Denoting by $(2.2) D_{k}$ the operator of covariant differentiation with respect to $\Gamma^{h}_{ji}$, we find that the curvature tensor $R_{kj} F_{ij}$ of $\Gamma^{h}_{ji}$ is of the form:

\[(2.9) \quad R_{kj} F_{ij} = (2.2) R_{kj} F_{ij} - \delta h_{j} p_{i} (2.2) D_{k} p_{i} + p^{h}_{k} p_{i},\]

From (2.4), (2.6) and (2.7), we easily obtain

\[(2.10) \quad (2.2) D_{k} p_{i} = F_{k} p_{i} - p_{k} p_{i} + \lambda g_{ki} + q_{k} q_{i} - p^{h}_{k} q_{i},\]

\[(2.11) \quad (2.2) \Gamma^{h}_{ji} - (2.2) \Gamma^{h}_{ij} = -2 F_{k} g_{ji} - \delta h_{k} p_{j} + \delta h_{j} p_{k} - p^{h}_{k} F^{h}_{j} p_{i} + p^{h}_{k} F^{h}_{j} - F^{h}_{j} g_{qi} + F^{h}_{j} q_{i}.\]

In the sequel, we introduced an $F$-connection $D$ whose components $\Gamma^{h}_{ji}$
are given by
\[ (2.12) \quad *\Gamma^h_{ji} = \Gamma^h_{ji} - \rho_j \delta^h_{pi} - \rho_j F^h_i - F^h_j q_i - \delta^h_{pi}, \]
where \( \Gamma^h_{ji} \) are the components of the complex Weyl-Hlavaty connection \( \Gamma^h \). Then (2.12) is written as
\[ (2.13) \quad *\Gamma^h_{ji} = \{ F^h_i \} + q_j F^h_i - \rho_j F^h_i - g_{ji} F^h_j - F_{ji} \]
by the help of (1.4).

Substituting (2.12) into (1.1), (1.2) and (1.3) and denoting by \( *D_i \) the covariant differentiation with respect to \( *\Gamma^h_{ji} \), we find
\[ (2.14) \quad *D_k g_{ji} = g_{kj} p_i + g_{ki} p_j + F_{kj} q_i + F_{ki} q_j, \]
\[ (2.15) \quad *D_k F_{ji} = g_{kj} q_i - g_{ki} q_j - F_{kj} p_i + F_{ki} p_j, \quad \text{(or} \quad *D_k F^h_j = 0). \]
In this case, we obtain
\[ (2.16) \quad *\Gamma^h_{ji} - *\Gamma^h_{ij} = -2 F_{ji} q^h + (q_j - \rho_j) F^h_i - (q_i - \rho_i) F^h_j, \]
and we call \( *D \) an F-connection which is closely related to the complex Weyl-Hlavaty connection.

Substituting (1.8), (2.3) and (2.5) into (2.9) successively, we obtain
\[ (2.17) \quad *R^h_{ki} = K^h_{ki} - g_{ji} (p^h_i - \frac{1}{2} \lambda \delta^h_j) + g_{ki} (p^h_j - \frac{1}{2} \lambda \delta^h_i) \]
-\( F_{ji} (q^h_k - \frac{1}{2} \lambda F^h_i) + F_{ki} (q^h_j - \frac{1}{2} \lambda F^h_j) \)
+\( (q_{kj} - q_{jk} - \lambda F_{kj}) F^h_i - (p_{kj} - p_{jk}) F^h_i \),
where \( p_{ji} \) and \( q_{ji} \) are defined by (1.9) and (1.10) respectively.

\( \text{§ 3. A characterization of a Kaehlerian manifold by the curvature tensor} \)
\( *R^h_{ki} \text{ defined by (2.17).} \)

In this section, we prove the following

**Theorem 3.1.** Let a real 2n-dimensional Kaehlerian manifold \( M(n \geq 2) \) admits the complex Weyl-Hlavaty connection. If the curvature tensor formed with the components of the connection \( *D \) defined by (2.13) vanishes locally, then \( M \) is of constant holomorphic sectional curvature locally, and vice-versa.

**Proof.** Assuming that
\[ (3.1) \quad *R^h_{ki} = 0, \]
we can write (2.17) in covariant form as
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\[ K_{kji} = g_{ji} \left( p_{kh} - \frac{1}{2} \lambda g_{kh} \right) - g_{ki} \left( p_{jh} - \frac{1}{2} \lambda g_{jh} \right) + F_{ji} \left( q_{kh} - \frac{1}{2} \lambda F_{kh} \right) \]

\[ -F_{ki} \left( q_{jh} - \frac{1}{2} \lambda F_{jh} \right) - (q_{kj} - q_{jk} - \lambda F_{kj}) F_{ih} + (p_{kj} - p_{jk}) F_{ih}. \]

(3.2)

For briefness, we now put

\[ P_{ji} = p_{ji} - \frac{1}{2} \lambda g_{ji} \]

and

\[ Q_{ji} = q_{ji} - \frac{1}{2} \lambda F_{ji}. \]

Then (3.2) becomes

\[ K_{kji} = g_{ji} P_{kh} - g_{ki} P_{jh} + F_{ji} Q_{kh} - F_{ki} Q_{jh} - (Q_{kj} - Q_{jk}) F_{ih} + (P_{kj} - P_{jk}) F_{ih}. \]

Transvecting (3.3) with \( F_{ki} \) and using (1.12), we find

\[ P_{ji} F_{k} = -Q_{jk}, \quad Q_{ji} F_{k} = P_{jk}. \]

Transvecting (3.5) with \( g^{kh} \), taking account of (3.6), and denoting by \( K_{ji} \) the Ricci tensor of \( M \), we obtain

\[ K_{ji} = P g_{ji} + Q F_{ji} + P_{ji} - Q_{li} F_{i} + P_{ij} F_{i} + Q_{ji}, \]

where we have put \( P = P_{ji} \) and \( Q = Q_{ji} \).

Transvecting (3.7) with \( F_{k} \), and taking account of (3.6), we find

\[ K_{ji} F_{i} = PF_{ij} + Q g_{ji} - Q_{ji} + Q_{ij} - P_{ij} + P_{ji}. \]

Substituting (3.8) into the well known equation \( K_{ji} F_{i} + K_{ji} F_{j} = 0 \), we obtain

\[ Q = 0. \]

Substituting (3.5) into the well known identity \( K_{kji} - K_{iinh} = 0 \) and transvecting it with \( g^{ih} \), we obtain

\[ P_{kj} - P_{jk} = 0. \]

by virtue of (3.6) and (3.9).

Substituting (3.5) into the identity \( K_{kji} = K_{iinh} \) and transvecting it with \( F^{kh} \), we find

\[ PF_{ji} + (2n - 1) Q_{ij} - Q_{ji} = 0. \]

Taking the symmetric part of (3.11), we obtain

\[ Q_{ji} + Q_{ij} = 0. \]
Substituting (3.12) into (3.11), we find

\[ Q_{ji} = \frac{P}{2n} P_{ji}. \]  

(3.13)

Substituting (3.13) into (3.6), we obtain

\[ P_{ji} = \frac{P}{2n} g_{ji}. \]  

(3.14)

Substituting (3.9), (3.13) and (3.14) into (3.7), we obtain

\[ K_{ji} = (1 + \frac{1}{n}) P g_{ji}, \]  

(3.15)

and from which, the scalar curvature \( K \) of \( M \) is given by

\[ K = 2(n+1)P. \]  

(3.16)

Substituting (3.15) and (3.16) into the identity \( \nabla_j K = 2\nabla_j K_j \), we obtain \( \nabla_j P = \frac{1}{n} (\nabla_j P) \delta_j^j \) and from which, \( \nabla_j P = 0 \), that is, \( P = \text{constant} \). Thus we obtain

\[ P = \frac{K}{2(n+1)}, \quad K = \text{constant}. \]  

(3.17)

Substituting (3.13), (3.14) and (3.17) into (3.5), we find

\[ K_{kji}^h = \frac{K}{2n(2n+1)} (g_{ji} \delta^k_h - g_{ki} \delta^j_h + F_{ji} F_{k}^h - F_{kj} F_{j}^h - 2 F_{kj} F_{i}^h), \]  

(3.18)

\( K \) being a constant.

Conversely, if the Riemann–Christoffel curvature tensor of \( M \) is of the form (3.18), then we consider the integrability condition of the differential equations

\[ p_i = \nabla_i p, \]  

(3.19)

\[ \nabla_j p_i = p_j p_i - q_j g_{ii} + \frac{K}{2n(2n+1)} g_{ji}, \]  

where \( K \) is a constant and

\[ q_j = -p_i F_{i}^j. \]  

(3.20)

By a straightforward computation, we see that the following equation is satisfied

\[ \nabla_k \nabla_j p_i - \nabla_j \nabla_k p_i = -K_{kji}^h p^h \]  

by virtue of (3.18).
Therefore a gradient covector field $p_i$ satisfying (3.19) is completely integrable locally. In this case, differentiating covariantly (3.20) and substituting (3.19) into it, we easily obtain

\begin{equation}
\nabla_j q_i = p_j q_i - q_j p_i + \frac{K}{2n(2n+1)} F_{ji},
\end{equation}

Substituting (3.19) and (3.21) into (1.9) and (1.10) respectively, we obtain

\begin{equation}
\rho_{ji} = \frac{\lambda}{2} g_{ji} = \frac{K}{2n(2n+1)} g_{ji},
\end{equation}

\begin{equation}
q_{ji} = -\frac{\lambda}{2} F_{ji} = \frac{K}{2n(2n+1)} F_{ji}.
\end{equation}

Substituting (3.22) and (3.23) into (2.17), we see that

\begin{equation}
*R_{kji}^h = 0
\end{equation}

by virtue of (3.18). Thus the theorem 3.1 is completely proved. (cf. [6])

II. A characterization of a cosymplectic manifold by a curvature tensor formed with a $\phi$-connection with torsion.

§ 1. Preliminaries.

We consider a cosymplectic manifold $M$ of real $2n+1$ dimensions covered by a coordinate neighborhoods $\{U; \gamma^h\}$ and denote by $g_{ji}$, $\phi^h$, $\xi^h$ and $\eta_j$ components of the Grayan metric tensor, those of the cosymplectic structure tensor, those of the cosymplectic vector and those of the cosymplectic $l$-form of $M$ respectively, where, here and in the sequel, the indices $h, i, j, \cdots$ run over the range $\{1, 2, \cdots, 2n+1\}$.

We denote by $K_{kji}^h$, $K_{ji}$ and $K$ the curvature tensor, the Ricci tensor and the scalar curvature of $M$ respectively.

Let $^D$ be an affine connection with torsion in a cosymplectic manifold $M$. We denote by $^D\Gamma_{kji}^h$ the components of the connection $^D$ and by $^D\Gamma_{ji}$ the operator of covariant differentiation with respect to $^D\Gamma_{kji}^h$.

If the affine connection $^D$ satisfies

\begin{equation}
^D_{kji} = -2 \rho_{kji},
\end{equation}

where $\gamma_{ji} = g_{ji} - \eta_j \eta_i$,

\begin{equation}
^D\phi_{ji} = -2 \rho_{ji} \phi_{ji} \quad \text{(or $^D\phi_{ji}^h = 0$)},
\end{equation}

\begin{equation}
^D\xi^h = 0 \quad \text{(or $^D\eta_j = 0$)},
\end{equation}

\begin{equation}
^D\phi^h = 0 \quad \text{(or $^D\phi^h = 0$)},
\end{equation}

\begin{equation}
^D\eta_j = 0 \quad \text{(or $^D\eta_j = 0$)},
\end{equation}

\begin{equation}
^D\xi^j = 0 \quad \text{(or $^D\xi^j = 0$)}.
\end{equation}
for a certain non-zero covector field $p_k$ and a vector field $u^h$, then $D'$ is called in [1] a cosymplectic Weyl-Hlavaty connection.

In a previous paper [1], we constructed an affine connection $D$ whose components $\Gamma_{ij}^h$ are given by $\Gamma_{ij}^h = \frac{1}{2} \gamma_{ij}^h$, that is,

$$\Gamma_{ij}^h = \{(h) + p_i p_j - p_h \gamma_{ij} + q_i q_j - \phi_{ij} q^h,$$

where $\{(h)$ are the Christoffel symbols of $M$ and

$$q_i = -p_i \phi_i^i, \quad p_i = q_i \phi_i^i, \quad \gamma_j^i = \gamma_{ij} g^i.$$

Denoting by $R_{kij}^h$ the curvature tensor of $\Gamma_{ij}^h$, we obtain (cf. [1])

$$R_{kij}^h = k_{kji}^h - \phi_{khi} q_{ji} + q_{ki} q_{ji} - \alpha_{khi} q_{gi} + \phi_{kji} q^h,$$

where

(1.6) $q_i = -p_i \phi_i^i, \quad p_i = q_i \phi_i^i, \quad \gamma_j^i = \gamma_{ij} g^i.$

(1.7) $R_{kij}^h = k_{kji}^h - \phi_{khi} q_{ji} + q_{ki} q_{ji} - \alpha_{khi} q_{gi} + \phi_{kji} q^h,$

$\lambda$ being defined by $\lambda = p_i p^i = q_i q^i$,

(1.8) $p_i = q_i \phi_i^i$,

(1.9) $q_i = -p_i \phi_i^i$,

(1.10) $\alpha_{ij} = -(\gamma_{ij} q_i - \gamma_{ij} q_j)$,

(1.11) $\beta_{ij} = 2(\gamma_{ij} q_i - \gamma_{ij} q_j)$

and $p_{k}^i = p_{k} g_{ih}, q_{k}^i = q_{k} g_{ih}, \beta_{k}^i = \beta_{k} g_{ih}$.

The following relations are easily checked.

(1.12) $p_{ji} = q_{ji} \phi_i^i, q_{ji} = -p_{ji} \phi_i^i$,

(1.13) $\alpha_{ji} = -(q_{ji} - q_{ij} - \lambda \phi_{ji}).$

§ 2. A certain $\phi$-connection which is closely related to the cosymplectic Weyl-Hlavaty connection.

In this section, we want to seek out a $\phi$-connection $D'$ in a cosymplectic manifold $M$ such that if the curvature tensor formed with the components of $D'$ vanishes then $M$ is of constant $\phi$-holomorphic sectional curvature.

For this aim, by the quite similar process to chapter I, § 2, we construct, in the sequel, a $\phi$-connection $D'$ whose components $\Gamma_{ij}^h$ are given by

$$\Gamma_{ij}^h = \gamma_{ij}^h - p_{ji} \phi_i^h - \phi_{j} q_{i} - \gamma_{ji} p_i,$$

where $\Gamma_{ij}^h$ are defined by (1.5), that is,
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\[ \Gamma^h_{ji} = \{ h^i_j \} + q_j \varphi_i^h - p_j \varphi_i^h - \gamma_{ji} p^h - \varphi_{ji} q^h. \]

Denoting by \( *D_j \) the operator of covariant differentiation with respect to \( \Gamma^h_{ji} \), we find

\[ *D_k g_{ji} = \gamma_{kj} p^j + \gamma_{ki} p_j + \varphi_{kj} q_i + \varphi_{ki} q_j, \]
\[ *D_k \varphi_{ji} = \gamma_{kj} q_j - \varphi_{kj} p_i + \varphi_{ki} p_j. \]

In this case, we obtain

\[ \Gamma^h_{ji} - \Gamma^h_{ij} = -2 \varphi_{ji} q^h + (q_j - p_j) \varphi_i^h - (q_i - p_i) \varphi_j^h, \]

and we call \( *D \) a \( \varphi \)-connection which is closely related to the cosymplectic Weyl-Hlavaty connection.

Computing the curvature tensor \( \ast R_{kji}^h \) of \( \Gamma^h_{ji} \) and using the notations (1.8), (1.9) and

\[ p^h_k = p_{ki} g^h_k, \quad q^h_k = q_{ki} g^h_k, \]

we obtain

\[ \ast R_{kji}^h = K_{kji}^h - \gamma_{ji} (p^h_k - \frac{\lambda}{2} \gamma^h_k) + \gamma_{ki} (p^h_j - \frac{\lambda}{2} \gamma^h_j) \]
\[ -\varphi_{ji} (q^h_k - \frac{\lambda}{2} \varphi^h_k) + \varphi_{ki} (q^h_j - \frac{\lambda}{2} \varphi^h_j) \]
\[ + (q_{kj} - q_{jk} - \lambda \varphi_{kj}) \varphi_i^h - (p_{kj} - p_{jk}) \varphi_i^h. \]

§3. A characterization of a cosymplectic manifold by the curvature tensor \( \ast R_{kji}^h \) defined by (2.6).

In this section, we prove the following

**Theorem 3.1** Let a real \((2n+1)\)-dimensional cosymplectic manifold \( M \) \((n \geq 2)\) admits the cosymplectic Weyl-Hlavaty connection. If the curvature tensor formed with the components of the connection \( *D \) defined by (2.2) vanishes locally, then \( M \) is of constant \( \varphi \)-holomorphic sectional curvature locally, and vice-versa.

**Proof.** Assuming that

\[ \ast R_{kji}^h = 0, \]

we can write (2.6) in the following covariant form:

\[ K_{kji}^h = \gamma_{ji} (p^h_k - \frac{\lambda}{2} \gamma^h_k) - \gamma_{ki} (p^h_j - \frac{\lambda}{2} \gamma^h_j) \]
\[ + \varphi_{ji} (q^h_k - \frac{\lambda}{2} \varphi^h_k) - \varphi_{ki} (q^h_j - \frac{\lambda}{2} \varphi^h_j). \]
For briefness, we now put

\[(3.3) \quad P_{ji} = p_{ji} - \frac{\lambda}{2} \gamma_{ji},\]

\[(3.4) \quad Q_{ji} = q_{ji} - \frac{\lambda}{2} \varphi_{ji}.\]

Then (3.2) becomes

\[(3.5) \quad K_{kjh} = \gamma_{ji} P_{kh} - \gamma_{ki} P_{jh} + \varphi_{ji} Q_{kh} - \varphi_{ki} Q_{jh} - (Q_{kj} - Q_{jk}) \varphi_{ih} + (P_{kj} - P_{jk}) \varphi_{ih}.\]

Transvecting (3.3) or (3.4) with \(\varphi_{ki}\) and using (1.12), we find

\[(3.6) \quad P_{ji} \varphi_{ki} = - Q_{ji}, \quad Q_{ji} \varphi_{ki} = P_{jk}.\]

From (3.6), we obtain

\[(3.7) \quad P_{ji} \tilde{e}^t = 0, \quad Q_{ji} \tilde{e}^t = 0.\]

Transvecting (3.5) with \(g^{kh}\) and taking account of (3.6) and (3.7), we obtain

\[(3.8) \quad K_{ji} = P_{ji} \gamma_{ji} + Q_{ji} \varphi_{ji} + P_{ji} + Q_{ji} - Q_{ij} \varphi_{ji} + P_{ij} \varphi_{ji},\]

where we have put \(P = P_{ji}^i\) and \(Q = Q_{ji}^i\).

Transvecting (3.8) with \(\varphi_{ki}\) and taking account of (3.6), we find

\[(3.9) \quad K_{ji} \varphi_{ki} = P \varphi_{ji} + Q \gamma_{ji} + P_{ji} - P_{ij} - Q_{ji} + Q_{ij}.\]

Substituting (3.9) into the well known equation

\[K_{ji} \varphi_{ki} + K_{it} \varphi_{ij} = 0,\]

we obtain

\[(3.10) \quad Q = 0.\]

Substituting (3.5) into the well known identity

\[(3.11) \quad K_{kjh} - K_{ihkj} = 0\]

and transvecting it with \(\gamma_{ik}\), we obtain

\[(3.12) \quad P_{jk} = P_{kj}\]

by virtue of (3.6) and (3.10).

Substituting (3.5) into (3.11) and transvecting it with \(\varphi_{kh}\), we find

\[(3.13) \quad P_{ji} + (2n+1) Q_{ij} - (Q_{tjf}^t + Q_{tjf}^l) - Q_{tjf}^t = 0.\]
On the other hand, taking account of (3.6), (3.7) and (3.12), we find

\[ P_{j}^{\xi} t = 0, \quad Q_{j}^{\xi} t = 0. \]

Then, (3.13) becomes

\[ P_{\varphi_{ji}} + (2n-1) Q_{ij} - Q_{ji} = 0. \]

Taking the skew-symmetric part of (3.15), we find

\[ Q_{ji} + Q_{ij} = 0. \]

Substituting (3.16) into (3.15), we obtain

\[ Q_{ji} = \frac{P}{2n} \varphi_{ji}. \]

Substituting (3.17) into (3.6), we obtain

\[ P_{ji} = \frac{P}{2n} \gamma_{ji}. \]

Substituting (3.10), (3.17) and (3.18) into (3.8), we find

\[ K_{ji} = (1 + \frac{1}{n}) P_{\gamma_{ji}}, \]

from which, we obtain

\[ K = 2(n+1) P. \]

Substituting (3.19) and (3.20) into the identity \( \mathcal{F} \mathcal{J} K = 2 \mathcal{F} \mathcal{J} K \), we obtain

\[ (1-n) \mathcal{F} \mathcal{J} P = \xi \mathcal{F} (\mathcal{F} \mathcal{J} P) \eta_{\mathcal{J}}. \]

Transvecting it with \( \xi \), we find \( \xi \mathcal{F} \mathcal{J} P = 0 \), by \( n \geq 2 \), from which \( \mathcal{F} \mathcal{J} P = 0 \), that is,

\[ P = \text{const.} = \frac{K}{2(n+1)}. \]

Substituting (3.17) and (3.18) into (3.5) and taking account of (3.20), we obtain

\[ K_{k}^{h j} = \frac{K}{4n(n+1)} [\gamma_{ji}^{h} \gamma_{k}^{j} - \gamma_{ki}^{h} \gamma_{j}^{k} + \varphi_{ji}^{h} \varphi_{k}^{j} - \varphi_{ki}^{h} \varphi_{j}^{k} - 2 \varphi_{kj}^{h} \varphi_{j}^{k}], \]

\( K \) being a constant.

Therefore \( M \) is of constant \( \varphi \)-holomorphic sectional curvature. (cf. [2]).

Conversely, if \( M \) is of constant \( \varphi \)-holomorphic sectional curvature, then the Riemann–Christoffel curvature tensor of \( M \) is of the form (3.21). In this case, we consider the integrability condition of the differential equations

\[ p_{i} = \mathcal{F} \mathcal{J} p \]

(3.22)
where $K$ is a constant and

$$q_j = -p_t \phi_j^i.$$  

By a straightforward computation, we see that the following equation is satisfied

$$\nabla_j \nabla_i p_i = -K_{kj}^i p_t$$

by the help of (1.6) and (3.21).

Therefore a gradient covector field $p_i$ satisfying (3.22) is completely integrable locally.

In this case, differentiating covariantly (3.23) and substituting (3.22) into it, we easily obtain

$$(3.24) \quad \nabla_j q_i - q_j \nabla_i p_i + \frac{K}{4n(n+1)} \phi_j^i.$$  

Substituting (3.22) and (3.24) into (1.8) and (1.9) respectively, we obtain

$$(3.25) \quad \frac{1}{2} \gamma^i_j = \frac{1}{4n(n+1)} \gamma^i_j,$$

$$(3.26) \quad \frac{1}{2} \phi_j^i = \frac{1}{4n(n+1)} \phi_j^i.$$  

Substituting (3.25) and (3.26) into (2.6), we see that

$$*R_{kij} = 0.$$  

Thus, the theorem 3.1 is completely proved.

III. A characterization of a Sasakian manifold by a curvature tensor formed with an $\eta$-connection with torsion.

§ 1. Preliminaries.

We consider a Sasakian manifold $M$ of real $2n+1$ dimensions covered by a coordinate neighborhoods \( \{ U ; y^h \} \) and denote by $g_{ij}$, $\phi_j^h$, $\xi^h$ and $\eta_j$ components of the Grayan metric tensor, those of the Sasakian structure tensor, those of the Sasakian vector and those of the Sasakian l-form of $M$ respectively, where, here and in the sequel, the indices $h, i, j, \cdots$ run over the range $\{1, 2, \cdots, 2n+1\}$.

We denote by $K_{kij}$, $K_{ij}$ and $K$ the curvature tensor, the Ricci tensor and
the scalar curvature of $M$ respectively.

In a previous paper [1], we considered an affine connection $\mathcal{D}$ satisfying (1.1)–(1.4) of chapter II in a Sasakian manifold $M$ and we called it a contact Weyl–Hlavaty connection.

Furthermore in [1], we also considered an affine connection $\mathcal{D}$ whose components $\Gamma_{ji}^k$ are related to the components $\Gamma_{ji}^k$ of $\mathcal{D}$ by $\Gamma_{ji}^k = \Gamma_{ji}^k - p_j \gamma_i$, where $\gamma_{ji} = g_{ji} - \eta_j \eta_i$, that is,

$$\Gamma_{ji}^k = \{_{ji}^k + \gamma_j^h p_i - \gamma_{ji}^k p_i + \varphi_j^k (q_i - \eta_i) + \eta_j^h (q_j - \eta_j) - \varphi_{ji} (q^k - \xi^k),$$

for a certain non-zero covector $p_i$, where $p^h = p_i g^{ih}$,

$$q_i = - p_i \varphi_i^l, \quad p_i = q_i \varphi_i^l, \quad \gamma_j^k = \gamma_{ji} g^{kh}, \quad q^k = q_i g^{kh}$$
in a Sasakian manifold $M$.

(Such a connection $\mathcal{D}$ is called “a special semi–symmetric metric $\varphi$-connection” in [5].)

Denoting by $R_{kji}^h$ the curvature tensor of $\Gamma_{ji}^k$, we obtain (cf. [1])

$$R_{kji}^h = K_{kji}^h - \gamma_j^k p_{ji} + \gamma_j^h p_{ki} - p_j^h \gamma_{ji} + p_j^h \gamma_{ki}$$

$$- \varphi_j^h q_{ji} + \varphi_j^h q_{ki} - q_j^h \varphi_{ji} + q_j^h \varphi_{ki}$$

$$- \alpha_{jk} \varphi_i^h - \varphi_{kj} \beta_i^h + (\varphi_k^h \varphi_{ji} - \varphi_j^h \varphi_{ki} - 2 \varphi_{kj} \varphi_i^h),$$

where

$$p_{ji} = p_j p_i - p_j p_i + (q_j - \eta_j) (q_i - \eta_i) + \frac{1}{2} \lambda \gamma_{ji},$$

$$q_{ji} = p_j q_i - p_i (q_j - \eta_j) - p_i (q_j - \eta_j) + \frac{1}{2} \lambda \varphi_{ji},$$

$\lambda$ being defined by $\lambda = p_i p^i = q_i q^i$,

$$\alpha_{ji} = - (V_j q_i - V_i q_j),$$

$$\beta_{ji} = \frac{1}{2} (p_j q_i - p_i q_j)$$

and $p_j^h = p_{ki} g^{ih}, \quad q_j^h = q_{ki} g^{ih}, \quad \beta_j^h = \beta_{ki} g^{ih}$.

The following relations are easily checked. (cf. [1])

$$p_{ji} \xi^i = \eta_j, \quad q_{ji} \xi^i = 0.$$

$$p_{ji} \varphi_i^l = - q_{ji}, \quad q_{ji} \varphi_i^l = p_{ji} - \gamma_j \eta_i.$$

$$\alpha_{ji} = - (q_{ji} - q_{ij} - \lambda \varphi_{ji}).$$

$$\alpha_{ji} \xi^l = q_{ji} \xi^l, \quad \beta_{ji} \xi^l = 0.$$
A certain $\eta$-connection which is closely related to the contact Weyl–Hlavaty connection.

In this section, we want to seek out an $\eta$-connection $^*D$ in a Sasakian manifold $M$ such that if the curvature tensor formed with the components of $^*D$ vanishes then $M$ is of constant $C$-holomorphic sectional curvature.

For this aim, by the quite similar process to chapter I, § 2, we construct, in the sequel, an $\eta$-connection $^*D$ whose connection $^*\Gamma_{ji}^h$ are given by

$$(2.1) \quad ^*\Gamma_{ji}^h=\Gamma_{ji}^h-\phi_j^h \gamma_i^h-\phi_i^h \gamma_j^h \psi_i,$$

where $\Gamma_{ji}^h$ are defined by (1.1), that is,

$$(2.2) \quad ^*\Gamma_{ji}^h=\{^h\}_i+\gamma^h_j \psi_j^i-\phi^h_j \psi_j^i-\phi^h_i \gamma^h_j,$$

and we call $^*D$ an $\eta$-connection which is closely related to the contact Weyl–Hlavaty connection.

Computing the curvature tensor $^*R_{kji}^h$ of $^*\Gamma_{ji}^h$ and using the notations (1.4), (1.5) and

$$(2.6) \quad ^*\Gamma_{ji}^h-^*\Gamma_{ij}^h=\gamma^h_i \psi_j^i-\gamma^h_j \psi_j^i-\phi^h_j \psi_j^i+\phi^h_i \gamma^h_j,$$

we obtain

$$(2.7) \quad ^*R_{kji}^h=K_{kji}^h-\gamma_{ji}^h(p^h_k-\frac{\lambda}{2}\gamma^h_j)+\gamma_{ki}^h(p^h_j-\frac{\lambda}{2}\gamma^h_j)$$

$\quad -\phi_{ji}^h[q^h_k-(1+\frac{\lambda}{2})\phi^h_k]+\phi_{ki}^h[q^h_j-(1+\frac{\lambda}{2})\phi^h_j]$
§ 3. A characterization of a Sasakian manifold by the curvature tensor \( \ast R_{kji}^h \) defined by (2.7).

In this section we prove the following

**Theorem 3.1.** Let a real \((2n+1)\)-dimensional Sasakian manifold \( M (n \geq 2) \) admits the contact Weyl-Hlavaty connection. If the curvature tensor formed with the components of the connection \( \ast D \) defined by (2.2) vanishes locally, then \( M \) is of constant \( \ast \)-holomorphic sectional curvature locally, and vice-versa.

**Proof.** Assuming that

\[
\ast R_{kji}^h = 0,
\]

we have from (2.7)

\[
K_{kji}^h = \gamma_{ji} P_{kj}^h - \gamma_{kj} P_{ji}^h + \varphi_{ji} Q_{kj}^h - \varphi_{kj} Q_{ji}^h - (Q_{kj} - Q_{jk}) \varphi_i^h
\]

\[
+ (P_{kj} - P_{jk}) \varphi_i^h - (\gamma_{ji} \eta_k - \gamma_{kj} \eta_i) \eta_i,
\]

where we have put

\[
P_{kji}^h = \frac{\lambda}{2} \gamma_{kj}, \quad P_{kj}^i = P_{kj} g^{ji}
\]

and

\[
Q_{kji} = q_{kj} - (1 + \frac{\lambda}{2}) \varphi_{kj}, \quad Q_{kj}^i = Q_{kj} g^{ji}.
\]

Transvecting (3.3) with \( \varphi_i^h \) and using (1.9), we obtain

\[
P_{kji} \varphi_j^i = -Q_{kj} - \varphi_{kj}.
\]

Similarly from (1.9) and (3.4), we obtain

\[
Q_{kji} \varphi_j^i = P_{kj} - g_{kj}.
\]

Transvecting (3.3) with \( \xi_j \) and using (1.8), we obtain

\[
P_{jkl} \xi^i = \eta_j.
\]

Similarly from (1.8) and (3.4), we obtain

\[
Q_{jkl} \xi^i = 0.
\]

Contracting with respect to \( k \) and \( h \) in (3.2) and taking account of (3.5) and (3.6), we obtain

\[
K_{jki} = (P - 2) \gamma_{ji} + (Q + 1) \varphi_{ji} + P_{ji} + Q_{ji} + P_{ij} \varphi_i^j - Q_{ij} \varphi_i^j + (2n - 1) \eta_j \eta_i,
\]

where we have put

\[
P = P_{ij}^i, \quad Q = Q_{ij}^j.
\]
Transvecting (3.9) with $\varphi^i_k$ and taking account of (3.5) and (3.6), we obtain

$$K_{ji}\varphi^i_k = -(P-1)\varphi_{ji} + (Q+1)\gamma_{ji} - Q_{ji} + P_{ji} - g_{ji}$$

(3.11)

Substituting (3.11) into the well known equation

$$K_{ji}\varphi^i_k + K_{it}\varphi^t_j = 0,$$

we have

$$2(Q+1)\gamma_{ji} - 2g_{ji} + (\eta_i P_{ij} + \eta_j P_{ji})\xi^t - (\eta_i Q_{ij} + \eta_j Q_{ji})\xi^t = 0.$$  

(3.12)

Transvecting (3.12) with $g^{ij}$, we see that

$$Q = 0.$$  

(3.13)

Substituting (3.13) into (3.12) and transvecting it with $\xi^t$, we obtain

$$P_{ij}\xi^t - Q_{ij}\xi^t = \gamma_{ji},$$

(3.14)

Substituting (3.2) into the well known identity

$$K_{kih} - K_{ihk} = 0$$

(3.15)

and transvecting it with $\gamma^{ih}$, we obtain

$$P_{kj} = P_{jk},$$

(3.16)

by the help of (3.5), (3.6) and (3.13).

Taking account of (3.7), (3.14) and (3.16), we see that

$$Q_{ij}\xi^t = 0.$$  

(3.17)

Substituting (3.2) into (3.15) and transvecting it with $\varphi^{kh}$, we obtain

$$(P-2n-1)\varphi_{ji} = Q_{ji} + (1-2n)Q_{ij}$$

(3.18)

by the help of (3.5), (3.6) and (3.16).

Since $\varphi_{ji} + \varphi_{ij} = 0$, we see from (3.18) that

$$Q_{kj} + Q_{jk} = 0.$$  

(3.19)

Therefore (3.18) is rewritten as

$$Q_{ji} = \left(\frac{P-1}{2n} - 1\right)\varphi_{ji}.$$  

(3.20)

Substituting (3.20) into (3.16), we obtain

$$P_{ji} = \frac{P-1}{2n}g_{ji} - \left(\frac{P-1}{2n} - 1\right)\eta_{ji}.$$  

(3.21)
On connections with torsions and those curvature tensored in Riemannian manifolds

On the other hand, if we take account of (3.5), (3.6), (3.13), (3.16) and (3.19), then (3.9) is rewritten as

\[(3.22) \quad K_{ji} = (P-1)g_{ji} + (2n+1-P)g_j\eta_i.\]

Transvecting (3.22) with $g^{ji}$ we easily see that

\[(3.23) \quad K = 2nP.\]

Substituting (3.22) and (3.23) into the well known identity $\nabla_j K = 2\nabla_j P$, we see that $(1-n)\nabla_j P = \xi^i(\nabla_i P)\eta_j$. Transvecting it with $\xi^i$, we find $\xi^i\nabla_i P = 0$ by the help of $n \geq 2$, from which $\nabla_j P = 0$, that is, $P = \text{const}$. Thus $M$ is an $\eta$-Einstein manifold ([3]).

Now we put

\[(3.24) \quad \frac{P-1}{2n} - 1 = k\]

$k$ being a constant.

Then, (3.20) and (3.21) are respectively rewritten as

\[(3.25) \quad Q_{ji} = k\varphi_{ji}.\]

\[(3.26) \quad P_{ji} = k\gamma_{ji} + g_{ji}.\]

Substituting (3.25) and (3.26) into (3.2) and recalling $\gamma_{ji} = g_{ji} - \eta_j\eta_i$, we obtain

\[(3.27) \quad K_{kji} = (k+1)(g_{kh}g_{ji} - g_{jh}g_{ki}) - k[\eta_h(g_{ji}\eta_h - g_{jh}\eta_i) - \eta_j(g_{ki}\eta_h - g_{kh}\eta_i) - \varphi_{ij}\varphi_{kh} + \varphi_{ik}\varphi_{jh} + 2\varphi_{ki}\varphi_{ih}].\]

Therefore $M$ is of a constant $C$-holomorphic sectional curvature ([3]).

Conversely, let the Riemann-Christoffel curvature tensor of $M$ be of the form (3.27). We consider the integrability condition of the differential equations

\[(3.28) \quad \nabla_i p = p_i.\]

\[(3.29) \quad \nabla_j p_i = (k+1)g_{ji} + p_j p_i - q_j q_i + q_j\eta_i + q_i\eta_j - (k+1)\eta_j\eta_i,\]

where

\[(3.30) \quad q_j = -p_i\varphi_{ji}.\]

By a straightforward computation, we see that the following equation is satisfied

\[\nabla_k \nabla_j p_i - \nabla_j \nabla_k p_i = -K_{kji}p_i\]

by the help of (1.2) and (3.27).
Therefore a gradient covector field $p_i$ satisfying (3.28) is completely integrable locally.

In this case, differentiating covariantly (3.29) and substituting (3.28) into it, we easily obtain

\[(3.30) \quad P_{ji} = p_j q_i + q_j p_i - \eta_j p_i - p_j \eta_i + (k+1) \varphi_{ji}.
\]

Substituting (3.28) and (3.30) into (1.4) and (1.5) respectively, we obtain

\[(3.31) \quad p_{ji} - \frac{\lambda}{2} \gamma_{ji} = (k+1) g_{ji} - k \eta_j \eta_i,
\]

\[(3.32) \quad q_{ji} - (1 + \frac{\lambda}{2}) \varphi_{ji} = k \varphi_{ji}.
\]

Substituting (3.31) and (3.32) into (2.7), we see that $* R_{kl}^{\ j^h} = 0$.

Thus, the theorem is completely proved.

References


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