1. It is well known that the derived normal ring of a noetherian integral domain is a Krull domain, which is a theorem of Mori-Nagata. J. Nishimura proved the theorem using properties of a complete local ring ([5]) and J. Querre proved the same theorem without using either completion or properties of a complete local ring ([7]). In this paper we obtain some properties of an ideal transform and the derived normal ring of a noetherian integral domain, which give another proof of the theorem.

2. In this paper all rings are commutative with identity and a local ring \((A, m)\) means a noetherian commutative ring \(A\) with only one maximal ideal \(m\). We denote the Krull dimension of a ring \(A\) by \(\dim A\).

**Proposition 1.** ([3], (33.2)) Let \(A\) be a noetherian integral domain with field of quotients \(K\), Let \(L\) be a finite algebraic extension of \(K\), and let \(B\) be a ring between \(A\) and \(L\). If \(\dim A=1\), then \(B\) is noetherian with \(\dim B \leq 1\) and for any non-zero element \(a\) in \(A\), \(B/aB\) is a finite \(A\)-module.

**Proof.** If \(\dim A=1\), it is well-known Krull-Akizuki's theorem and so we omit the proof.

Let \(I\) be an ideal of a ring \(A\), and let \(Q(A)\) be the total quotient ring of \(A\). \(A(I) = \{x \in Q(A) : xI^n \subseteq A\text{ for some natural number }n\}\) is called the ideal transform of \(A\) with respect to \(I\). We denote the derived normal ring of an integral domain \(A\) by \(\overline{A}\), which is the integral closure of \(A\) in its field of quotients.

**Proposition 2.** Let \((A, m)\) be a local domain and \(\dim A \geq 2\). Then the following conditions are equivalent: (1) \(A(m) = A\), (2) \(A(m) \cap \overline{A} = A\).

**Proof.** Since \(\dim A \geq 2\), we have \(xm \subseteq m\) for \(x \in A : m\), and it follows that \(A(m) \neq A\) implies \(A(m) \cap \overline{A} \neq A\).

**Proposition 3.** Let \((A, m)\) be a local domain and \(\dim A = n \geq 2\). Suppose for any noetherian integral domain \(B\) with \(\dim B < n\), \(B\) is a Krull domain.
Then the following conditions are equivalent:
(1) \( A(m) \subseteq \overline{A} \), (2) \( \overline{A} \) has no maximal ideal of height 1.

Proof. In view of Proposition (1.2) in \([6]\) it follows that (2) implies (1). Now we shall prove (1) implies (2). By definition \( A(m) = \bigcap_{i=1}^{r} A_{f_i} \), where \( m = f_1 A + \cdots + f_r A \). Since \( \dim A_{f_i} < \dim A = n \), by hypothesis \( A_{f_i}, i = 1, \ldots, r \), is a Krull domain. Hence \( \overline{A} = \overline{A(m)} = \bigcap_{i=1}^{r} \overline{A_{f_i}} \) is a Krull domain.

Now suppose there is a maximal ideal \( \mathfrak{m} \) of \( \overline{A} \) with \( \text{ht} \mathfrak{m} = 1 \). Then for non-zero element \( a \) in \( \mathfrak{m} \), \( a \overline{A} = \mathfrak{m}_a \overline{A} \cap \overline{A_{f_1}} \cap \cdots \cap \overline{A_{f_r}} \). Since \( \mathfrak{m}_a + I = \overline{A} \) with \( I = \overline{f_2(e_2)} \cap \cdots \cap \overline{f_r(e_r)} \), we have \( b + c = 1 \) for some \( b \in \mathfrak{m}_a \) and \( c \in I \).

Suppose there exist two distinct maximal ideals \( \mathfrak{m} \) and \( \mathfrak{n} \) in \( \overline{A} \) such that \( \mathfrak{m}' = \mathfrak{m} \cap A[b] = \mathfrak{n} \cap A[b] \). Then, as \( I \subseteq \mathfrak{n} \), \( c \in \mathfrak{n} \) and \( b \in \mathfrak{m} \). On the other hand \( b \in \mathfrak{m}_a \cap A[b] \subseteq \mathfrak{n} \), which is a contradiction. Hence \( \mathfrak{m} \) is the only one maximal ideal of \( \overline{A} \) which lies over \( \mathfrak{m}' \).

Let \( A_1 = A[b] \), \( \text{rad} A_1 = \mathfrak{m} \), and \( \mathfrak{m}' = \mathfrak{m} \cap A_1 \). Then, since \( A(m) \) is noetherian \([1]\), \( A_1(\mathfrak{m}) \) is a finite \( A(m) \)-module, and \( A_1(\mathfrak{m}) \subseteq \overline{A(m)} = \overline{A} = A_1 \). Hence \( \overline{A} \) is a Krull domain. Therefore \( \overline{A} \) has no maximal ideal of height 1.

Proposition 4. Let \((A, m)\) be a local domain and \( B = A(m) \cap \overline{A} \). Then the set of maximal ideals of \( A \) with height 1 and the set of maximal ideals of \( B \) with height 1 are in one to one correspondence.

Proof. \((B, n_1, \ldots, n_r)\) is a semi-local domain with Jacobson radical \( n = n_1 \cap \cdots \cap n_r \) \([1]\). Suppose that \( n_1, \ldots, n_\alpha \) are maximal ideals of height 1 and that \( n_{\alpha+1}, \ldots, n_r \) are maximal ideals of height >1. Then \( B(n) = \bigcap_{j=\alpha}^{r} B_{n_j} = B_{n_j} \), where \( T = B - \bigcup_{j=\alpha}^{r} n_j \). Since \( B(n) \cap \overline{B} = A(m) \cap \overline{A} = B, B_{n_j} = (B(n) \cap \overline{B})_{n_j} = B_{n_j} \cap \overline{B}_n \) for \( j > \alpha \). Hence by Proposition 2, \( B_{n_j} = B_{n_j} \cap \overline{B}_n \) for \( j > \alpha \) and \( B_{n_\alpha} = B \cap B_{n_\alpha} \). Thus by Proposition 3, \( \overline{B}_T \) has no maximal ideal of height 1. Therefore if \( \mathfrak{m} \) is a maximal ideal of \( \overline{A} \) with \( \text{ht} \mathfrak{m} = 1 \) and if \( n = \mathfrak{m} \cap B \), then \( \text{ht} n = 1 \).

Now let \( S_i = B - n_i \) for \( i \leq \alpha \). Then \( B_{S_i} = (A(m) \cap \overline{A})_{S_i} = A(m)_{S_i} \cap \overline{A} = A_{S_i} \). Hence \( \mathfrak{m} = n_i B_{S_i} \cap \overline{A} \) is the only maximal ideal of \( \overline{A} \) which lies over \( n_i \).

Proposition 5. Let \((A, m)\) be a local domain with \( \dim A = n \geq 2 \). Suppose that \( \overline{A} \) is a Krull domain for any noetherian integral domain \( R \) of dimension <\( n \). Then \( \overline{A} \) is a Krull domain.

Proof. Let \( B = A(m) \cap \overline{A} \). Then \((B, n_1, \ldots, n_r)\) is a semi-local domain. Let \( n_i, 1 \leq i \leq \alpha \), be maximal ideals of height 1, and let \( n_j, j > \alpha \), be maximal
ideals of height>1. Then $E = (\bigcap_{i=1}^{\alpha} B_{n_i}) \cap (\bigcap_{j=\alpha+1}^{r} B_{n_j})$, where $B_{n_i}$, $1 \leq i \leq \alpha$, is a discrete valuation ring by Proposition 4. For $j > \alpha$, $B_{n_j}(n_j) \subseteq B_{n_j}$ and $B_{n_j}(n_j) = \bigcap_{k=1}^{\alpha+1} (B_{n_j})_{b_{jk}}$, where $n_j = (b_{j1}, \ldots, b_{j\alpha})$. Hence $B_{n_j} = B_{n_j}(n_j) = \bigcap_{k=1}^{\alpha+1} (B_{n_j})_{b_{jk}}$. Since $\text{dim}(B_{n_j})_{b_{jk}} \leq n-1$, by hypothesis $(B_{n_j})_{b_{jk}}$ is a Krull domain and hence $\bar{A} = \bar{B}$ is a Krull domain.

**Proposition 6.** Let $A$ be a noetherian integral domain with $\text{dim} A = n \geq 2$. Suppose that $\bar{R}$ is a Krull domain for any local domain $R$ with $\text{dim} R \leq n$. Then $\bar{A}$ is a Krull domain.

**Proof.** $\bar{A} = \bigcap_{m} \bar{A}_m$, where the intersection runs through all maximal ideals of $A$. By hypothesis each $\bar{A}_m$ is a Krull domain, so it is sufficient to show that for any non-zero element $a$ in $A$, $\mathcal{F} = \{\bar{p} \in \text{Spec}(\bar{A}) : \text{ht } \bar{p} = 1, a \in \bar{p}\}$ is a finite set. Thus we shall prove the following:

1. $\mathcal{F}_0 = \{\bar{p} : \bar{p} \cap A = \bar{p}, \bar{p} \in \mathcal{F}\}$ is finite.
2. For each $\bar{p} \in \mathcal{F}_0$, $\mathcal{F}_p = \{\bar{p} \in \mathcal{F} : \bar{p} \cap A = \bar{p}\}$ is finite.

**Proof of (2).** The number of elements in $\mathcal{F}_p$ is the number of maximal ideals in $\bar{A}_p$ with height 1 which lies over the maximal ideal of the local domain $A_p$. Let $B = A_p(\bar{p}) \cap \bar{A}_p$. Then $B$ is a semi-local domain (\cite{1}). Hence $\mathcal{F}_p$ is a finite set by Proposition 4.

**Proof of (1).** Let $\bar{p} = \bar{p} \cap A, \bar{p} \in \mathcal{F}$. If $\text{ht } \bar{p} = 1$, then $\bar{p} \in \text{Ass}(aA)$. Thus the number of $\bar{p}$ with $\text{ht } \bar{p} = 1$ is finite. Now suppose $\text{ht } \bar{p} > 1$. If $A_p(\bar{p}) = A_p$, then $A_p(\bar{p}) \subseteq A_p$. Hence $\bar{A}_p$ has no maximal ideal of height 1 by Proposition 3. This is a contradiction. Therefore $A_p(\bar{p}) \supseteq A_p$. Now we need a lemma.

**Lemma.** Let $(A, m)$ be a local domain with field of quotients $K$. If $A(m) \supseteq A$ then for any $a \neq b$ in $m, \bar{m}$ is an associated prime divisor of $bA$.

**Proof of Lemma.** Let $m = (a_1, \ldots, a_r) = (a_1, \ldots, a_r, b)$. Then $A(m) = \bigcap_{j=1}^{r} A_{a_j} \cap A_b = A$. Hence there is an element $x$ in $A(m)$ but $x \notin A$. We may assume $xm \subseteq A$. Write $x = d/bi$. Then, as $xm \subseteq A$, $dm \subseteq b_i A$ i.e., $b_i : d \supseteq m$.

If $b_i : d \neq m$, then $b_i : d = A$ and $d = b_i a$ for some $a$ in $A$. Hence $x \in A$, which is a contradiction. Thus $b_i : d = m$. Now consider $b_i^{-1} : d \supseteq m$. Then it follows that either there is an element $c$ in $A$ such that $b : c = m$ or $b_i^{-1} : d = m$. Consequently, we have $b : c = m$ for some $c$ in $A$. Lemma is proved.

By Lemma $\mathcal{F}_p$ is an associated prime ideal of $aA$ but the number of associated prime ideals of $aA$ is finite. Hence $\mathcal{F}_0$ is finite.
PROPOSITION 7. ([3], (33.10)) The derived normal ring $\overline{A}$ of a noetherian integral domain $A$ is a Krull domain.

Proof. In view of Propositions 1, 5 and 6 the assertion is true for any noetherian domain of finite Krull dimension.

For a general noetherian domain $A$, since $\dim A_m < \infty$ for any maximal ideal $m$ and $\overline{A} = \cap_m A_m$, to get the assertion, it is sufficient to show that for any $0 \neq a$ in $A$, $\mathcal{I} = \{\overline{\mathfrak{p}} \in \text{Spec}(\overline{A}) : \text{ht} \overline{\mathfrak{p}} = 1, a \in \overline{\mathfrak{p}}\}$ is a finite set. But the same reasoning as in Proposition 6 gives the claim.

PROPOSITION 8. ([3], (33.12)) The derived normal ring $\overline{A}$ of a noetherian integral domain $A$ of Krull dimension $\leq 2$ is again noetherian.

Proof. If $\dim A \leq 1$, the assertion is clear by Proposition 1. Suppose $\dim A = 2$. Let $\overline{\mathfrak{p}}$ be a prime ideal of $\overline{A}$ with height 1 and let $\mathfrak{p} = \mathfrak{p} \cap A$. If $\overline{\mathfrak{p}}$ is not maximal, then $\text{ht} \mathfrak{p} = 1$. Since the quotient field $K(\overline{A}/\overline{\mathfrak{p}})$ of $\overline{A}/\overline{\mathfrak{p}}$ is a finite algebraic extension of the quotient field $K(\mathfrak{p})$ of $A/\mathfrak{p}$ and $\dim (A/\mathfrak{p}) = 1$, by Proposition 1 $\overline{A}/\overline{\mathfrak{p}}$ is noetherian. Thus by Mori-Nishimura’s theorem ([4], Theorem), $\overline{A}$ is noetherian.

Note that in Proposition 1 $B$ is not necessarily finite ([3], p. 205, Example 3). In Proposition 8 if $B$ is a ring between $A$ and $\overline{A}$, then $B$ is not necessarily noetherian ([3], p. 207, Example 4), and in Proposition 7 even $A$ is not necessarily noetherian ([3], p. 207, Example 5).

The author expresses his hearty thanks to Professor M. Nagata and Dr. J. Nishimura for their valuable suggestions.

References


Jeonbug National University