§1. Introduction

1.1 Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space, $\mathcal{L}(\mathcal{H})$ the algebra of all bounded operators $\mathcal{H}$, $\mathcal{K}(\mathcal{H})$ the ideal of compact operators, $\mathcal{O}(\mathcal{H})$ the quotient algebra $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, and $\pi$ the canonical homomorphism of $\mathcal{L}(\mathcal{H})$ onto $\mathcal{O}(\mathcal{H})$. Let $X$ be a compact Hausdorff space and $C(X)$ be the algebra of all complex-valued continuous functions on $X$. Brown, Douglas and Fillmore initiated in their pioneer work $[2]$ the study of the group $\text{Ext} X$ consisting of unitary equivalence classes of unital $*$-monomorphism $\tau : C(X) \to \mathcal{O}(\mathcal{H})$. The group $\text{Ext} X$ has many interesting features. Recently the theory of $\text{Ext}$ has been generalized to non-commutative $C^*$-algebras with remarkable successes by work of many mathematicians (see for example $[1]$).

1.2 Let $\mathcal{M}$ be a $\text{II}_\infty$-factor acting on $H$. It is well-known that $\mathcal{M}$ possesses an ideal analogous to $K(H)$; namely, the norm-closed two sided ideal $\mathcal{K}(\mathcal{M})$ generated by all finite projections in $\mathcal{M}$. Let $\mathcal{O}(\mathcal{M})$ denote the quotient algebra and $\pi$ the canonical homomorphism. Then one can consider unitary equivalence classes of unital $*$-monomorphism $\tau : C(X) \to \mathcal{O}(\mathcal{M})$ and can ask whether there exists a parallel theory in the context of $\text{II}_\infty$-factors. Fillmore $[5]$ and Cho $[4]$ indeed succeeded in developing extension theory relative to $\text{II}_\infty$-factors. Then the obvious question would be: is there any fruitful theory of extension relative to a $\text{II}_\infty$-factor for separable nuclear $C^*$-algebras? Arveson’s observation and Choi–Effros lifting theorem of completely positive maps (see for example $[1]$) make it possible that the unitary equivalence class of unital $*$-monomorphism $\tau : \mathcal{A} \to \mathcal{O}(\mathcal{M})$ for a separable nuclear $C^*$-algebra $\mathcal{A}$ forms an abelian group provided that one can have the Voiculescu’s non-commutative Weyl–von Neumann theorem analogue in the context of $\text{II}_\infty$-factors.

1.3 In this note, we examine lifting problem of unital $*$-monomorphism $\tau : \mathcal{A} \to \mathcal{C}(\mathcal{M})$. Similar lifting problem for the classical Calkin algebra was
considered earlier by Thayer [6]. Our result says that for UHF algebras \( \mathcal{A} \) any unital \(*\)-monomorphism \( \tau: \mathcal{A} \to \mathcal{O}(\mathcal{M}) \) can be lifted to unital \(*\)-monomorphism \( \sigma \) such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\sigma} & \mathcal{M} \\
\text{\Leftrightarrow} & \downarrow \tau & \text{\Leftrightarrow} \\
\mathcal{O}(\mathcal{M})
\end{array}
\]

§ 2. Liftings for finite-dimensional \( C^* \)-algebra

2.1 Let \( \mathcal{A} \) be a \( C^* \)-algebra. A family of partial isometries \( \{e_{ij}\}_{i,j=1}^n \) of \( \mathcal{A} \) such that

\[
e_{ij}e_{km} = 0 \text{ for } j \neq k
\]

\[
e_{ij}e_{km} = e_{im} \text{ for } j = k
\]

\[
e_{ij}^* = e_{ji}
\]

is called a system of matrix units in \( \mathcal{A} \).

For example if \( \mathcal{A} \) is a full matrix algebra \( \mathcal{M}_n \), then the matrices \( e_{ij} \) having entries zero except in the \((i, j)\)-position and having 1 in the \((i, j)\)-position forms a matrix units in \( \mathcal{M}_n \). Let \( \{u_{ij}\}_{i,j=1}^n \) be a matrix units in \( \mathcal{T}(\mathcal{W}) \). Then it is easy to see that any system of matrix units \( \{e_{ij}\} \) in \( \mathcal{M}_n \) determines a unique \(*\)-monomorphism \( \tau: \mathcal{M}_n \to \mathcal{O}(\mathcal{M}) \) such that \( \tau(e_{ij}) = u_{ij} \) for all \( i, j \leq 1, 2, \ldots, n \). In order to find \(*\)-monomorphism \( \sigma: \mathcal{M}_n \to \mathcal{M} \) such that \( \tau = \pi \circ \sigma \), we begin with the following lemma, which is type \( II_0 \) version of Calkin’s lifting theorem.

2.2 Lemma. Suppose that \( p \) and \( q \) are projections in \( \mathcal{O}(\mathcal{M}) \) such that \( p \leq q \). Then there exist projections \( P \) and \( Q \) in \( \mathcal{M} \) such that \( P \leq Q \) and \( \pi(P) = p, \pi(Q) = q \).

Proof. First we will prove that for any projection \( q \) in \( \mathcal{O}(\mathcal{M}) \) there exists a projection \( Q \) in \( \mathcal{M} \) with \( \pi(Q) = q \). To prove this, we choose \( X \) in \( \mathcal{M} \) such that \( \pi(X) = q \). Since \( \pi(X + X^*)/2 = q \), we may and do assume that \( X \) is a self-adjoint element of \( \mathcal{M} \). Let \( Q \) be the spectral projection of \( X \) corresponding to the interval \((\frac{1}{2}, \infty)\). Then the same argument as in [2, Theorem 2.4] tells us that \( X = Q \) is in the compact ideal \( \mathcal{K}(\mathcal{M}) \). Hence \( \pi(Q) = q \). Since \( p \leq q \), \( p \) belongs to \( q \mathcal{O}(\mathcal{M})q \). But \( \pi(q \mathcal{O}(\mathcal{M})q) = q \mathcal{O}(\mathcal{M})q \) and hence there exists a \( p \) in \( q \mathcal{O}(\mathcal{M})q \) such that \( \pi(P) = p \). This completes the proof.

2.3 Lemma. Suppose that \( P \) and \( Q \) are orthogonal projections in \( \mathcal{M} \). Let \( U \) be a partial isometry in \( \mathcal{O}(\mathcal{M}) \) such that \( u^* u = \pi(P), \ uu^* = \pi(Q) \). Then there
exists a partial isometry $U$ in $\mathcal{M}$ with $\pi(U) = u$, $U^*U \leq P$ and $UU^* \leq Q$.

Proof. Choose $X$ in $\mathcal{M}$ with $\pi(X) = u$. Let $\pi(Q) = q = \pi(P)$, $\pi(QP) = q\pi(X) = qu = qU^*U \leq P$. Let $X = V(X^*X)^{1/2}$ be its polar decomposition. Since $\pi((X^*X)^{1/2}) = (u^*u)^{1/2} = \pi(P)$, by 2.2 there exists a projection $E$ in $\mathcal{M}$ with $\pi(E) = \pi(P)$, namely the spectral projection of $(X^*X)^{1/2}$ corresponding to $(\frac{1}{2}, \infty)$. Moreover $E$ is a subprojection of the range projection of $(X^*X)^{1/2}$. Thus $E$ is a subprojection of the initial projection of the partial isometry $V$. Hence $VE$ is a partial isometry with the desired properties.

2.4 Lemma. Let $\{e_{ij}\}_{i,j=1}^{n}$ be a system of matrix units in $\mathcal{O}(\mathcal{M})$. Then there exists a system of matrix units $\{E_{ij}\}_{i,j=1}^{n}$ in $\mathcal{M}$ such that $\pi(E_{ij}) = e_{ij}$ for all $i, j = 1, 2, \ldots, n$.

Proof. We will prove this by the principle of mathematical induction. For $n=2$, by 2.2 we can choose two orthogonal projections $F_{11}$ and $F_{22}$ in $\mathcal{M}$ with $\pi(F_{11}) = e_{11}$ and $\pi(F_{22}) = e_{22}$. By 2.3 there exists a partial isometry $E_{21}$ in $\mathcal{M}$ such that $\pi(E_{21}) = e_{21}$, $E_{21}^*E_{21} \leq F_{11}$, $E_{21}E_{21}^* \leq F_{22}$. Since $F_{11} - E_{21}^*E_{21}$ and $F_{22} - E_{21}E_{21}^*$ are finite projections, we have $\pi(F_{11}) = \pi(E_{21}^*E_{21})$ and $\pi(F_{22}) = \pi(E_{21}E_{21}^*)$. Finally we put $E_{11} = E_{21}^*E_{21}$ and $E_{22} = E_{21}E_{21}^*$. Then $\{E_{ij}\}_{1 \leq i, j \leq 2}$ is the desired system of matrix units. Now suppose that a system of matrix units $\{F_{ij}\}_{2 \leq i, j \leq n}$ has been chosen so that $\pi(F_{ij}) = e_{ij}$, $2 \leq i, j \leq n$. Choose a projection $F_{11}$ which is orthogonal to $F_{22} + F_{33} + \ldots + F_{nn}$ (possible by the proof of 2.2). Apply 2.3 to get a partial isometry $E_{21}$ such that $\pi(E_{21}) = e_{21}$, $E_{21}^*E_{21} \leq F_{11}$, $E_{21}E_{21}^* \leq F_{22}$. We put $E_{11} = E_{21}^*E_{21}$, $E_{22} = E_{21}E_{21}^*$. Then $\pi(E_{11}) = \pi(F_{11})$ and $\pi(E_{22}) = \pi(F_{22})$. We put $E_{ii} = F_{ii} - E_{21}F_{2i}E_{21}$, for $i = 3, 4, \ldots, n$. Then $\{E_{ii}\}_{2 \leq i \leq n}$, will generate the desired system of matrix units.

2.5 Let $P$ be a finite projection in $\mathcal{L}(\mathcal{K})$. Let $n$ be a natural number. Then there exist mutually orthogonal equivalent projections $P_1, P_2, \ldots, P_n$ such that $P_1 + P_2 + \ldots + P_n = P$ if and only if $n$ divides the vector space dimension of the range space of $P$.

However, in a II$_\infty$-factor $\mathcal{M}$ for any finite projection $P$ and for any natural number $n$ there exist mutually orthogonal equivalent projections $P_1, P_2, \ldots, P_n$ such that $P_1 + P_2 + \ldots + P_n = P$. This distinction makes the following theorem possible.

2.6 Theorem. For any unital *-monomorphism $\tau : M_n \to \mathcal{O}(\mathcal{M})$, there exists a unital *-monomorphism $\sigma : M_n \to \mathcal{M}$ such that $\tau = \pi \circ \sigma$. 

Proof. Let \( \{e_{ij}\}_{i,j=1}^{n} \) be a system of matrix units for \( M_n \). It suffices to show that there exists a system of matrix units \( \{F_{ij}\} \) in \( \mathcal{M} \) such that \( \pi(F_{ij}) = \tau(e_{ij}) \) and \( F_{11} + F_{22} + \cdots + F_{nn} = 1 \). Let \( E_{ij} \) be a system of matrix units in \( \mathcal{M} \) which lifts \( \tau(e_{ij}) \) (it is possible by 2.4). Since \( \pi(E_{11} + \cdots + E_{nn}) = 1 \), the projection \( P = 1 - (E_{11} + \cdots + E_{nn}) \) is finite in \( \mathcal{M} \). Choose mutually equivalent orthogonal projections \( P_1, P_2, \ldots, P_n \) such that \( P_1 + P_2 + \cdots + P_n = P \). Let \( U_{11} \) be a partial isometry connecting \( P_1 \) to \( P_i \). Note that \( P_i \) and \( U_{11} \) are compact elements in \( \mathcal{M} \). Set \( F_{ii} = E_{ii} + P_i \) and \( F_{i1} = E_{i1} + U_{11} \). Then \( \{F_{ij}\}_{2 \leq i \leq n} \) will generate a system of matrix units \( \{F_{ij}\} \) with \( F_{11} + F_{22} + \cdots + F_{nn} = 1 \). This completes the proof.

Since any finite dimensional C*-algebra is a direct sum of full matrix algebras, we get:

2.7 Corollary. Let \( \mathcal{A} \) be a finite dimensional C*-algebra. Let \( \tau : \mathcal{A} \to \mathcal{B}(\mathcal{M}) \) be a unital *-monomorphism. Then there exists a unital *-monomorphism \( \sigma : \mathcal{A} \to \mathcal{M} \) such that \( \tau = \pi \circ \sigma \).

2.8 Lemma. Let \( \mathcal{A} \) be a full matrix algebra. Suppose that \( \sigma_1 \) and \( \sigma_2 \) are unital *-monomorphisms of \( \mathcal{A} \) into \( \mathcal{M} \). Then there exists a unitary \( U \) in \( \mathcal{M} \) such that \( \sigma_1(x) = U^* \sigma_2(x) U \) for all \( x \) in \( \mathcal{A} \).

Proof. Let \( \{E_{ij}\}_{1 \leq i, j \leq n} \) be a system of matrix units for \( \mathcal{A} \). We put \( e_{ij} = \sigma_1(E_{ij}) \) and \( f_{ij} = \sigma_2(E_{ij}) \). It suffices to show that there exists a unitary \( U \) in \( \mathcal{A} \) such that \( e_{ij} = U^* f_{ij} U \) for all \( 1 \leq i, j \leq n \). To this end, since \( \mathcal{M} \) is a \( \sigma \)-finite II\(_{\infty} \)-factor, the infinite projections \( e_{11} \) and \( f_{11} \) are equivalent. Hence there exists an element \( V \) of \( \mathcal{M} \) such that \( V^* V = e_{11} \) and \( V V^* = f_{11} \). We put \( U = \sum_{i=1}^{n} f_{11} V e_{1i} \). Then it is easy to check that \( e_{ij} = U^* f_{ij} U \). This completes the proof.

2.9 Corollary. Let \( \mathcal{A} \) be a finite dimensional C*-algebra. Let \( \tau_1, \tau_2 : \mathcal{A} \to \mathcal{B}(\mathcal{M}) \) be unital *-monomorphisms. Then there exists a unitary \( U \) in \( \mathcal{M} \) such that

\[
\tau_1(x) = \pi(U)^* \tau_2(x) \pi(U)
\]

for all \( x \) in \( \mathcal{A} \).

Proof. According to Corollary 2.7, each \( \tau_i \) has a unital lifting \( \sigma_i \) satisfying \( \tau_i = \pi \circ \sigma_i \), \( i = 1, 2 \). Application of Lemma 2.8 to summand by summand will give us a unitary \( U \) implementing the requirement.

2.10 Remark. Two extensions (i.e., unital *-monomorphisms) \( \tau_1, \tau_2 : \mathcal{A} \to \mathcal{B}(\mathcal{M}) \) are said to equivalent if there exists a unitary \( U \) in \( \mathcal{M} \) such that \( \tau_1(x) = \pi(U)^* \tau_2(x) \pi(U) \) for all \( x \) in \( \mathcal{A} \). The sum of \( \tau_1 + \tau_2 \) is the extension \( \tau : \mathcal{A} \to \mathcal{B}(\mathcal{M}) \) defined as follows: choose isometries \( V_1 \) and \( V_2 \) in \( \mathcal{M} \) such
that $V_1 V_1^* + V_2 V_2^* = 1$, and let $$(\tau_1 + \tau_2)(x) = \pi(V_1) \tau_1(x) \pi(V_1^*) + \pi(V_2) \tau_2(x) \pi(V_2^*)$$ for all $x$ in $\mathcal{A}$. The equivalence class of $\tau_1 + \tau_2$ is independent of the choice of isometries in the definition. Let $\text{Ext}^* \mathcal{A}$ denote the equivalence classes of extensions. Then for commutative $C^*$-algebra $\mathcal{A}$, $\text{Ext}^* \mathcal{A}$ is an abelian group (see [4] for details). Corollary 2.9 can be restated as follows: For finite dimensional $C^*$-algebra $\mathcal{A}$, the $\text{Ext}^* \mathcal{A}$ is always trivial group.

We close this section with the following.

2.11 Theorem. Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are full matrix algebras and that $\mathcal{A}_1$ is a subalgebra of $\mathcal{A}_2$ with the same unit. If $\tau_1, \tau_2 : \mathcal{A}_2 \to \mathcal{M}(\mathbb{C})$ is a unital $*$-monomorphism and $\sigma_1 : \mathcal{A}_1 \to \mathcal{M}$ is a unital $*$-monomorphism such that $\pi \circ \sigma_1 = \tau_1$, and then $\tau_2 / \mathcal{A}_1 = \tau_1$ there exists a unital $*$-monomorphism $\sigma_2 : \mathcal{A}_2 \to \mathcal{M}$ such that $\pi \circ \sigma_2 = \tau_2$ and $\sigma_2 |_{\mathcal{A}_1} = \sigma_1$.

Proof. Let $\{e_{ij}\}_{i,j=1}^{n}$ be a system of matrix units for $\mathcal{A}_1$ and $\{f_{ij}\}_{i,j,k}$ be a system of matrix units for $\mathcal{A}_2$. Since $\mathcal{A}_1 \subset \mathcal{A}_2$, $n$ divides $m$. Let $m = kn$. By rearranging $f_{ij}$ if necessary, we can assume that $e_{11} = f_{11} + f_{22} + \ldots + f_{kk}$, $e_{22} = f_{k+1,k+1} + \ldots + f_{2k,2k}$, ..., By applying 2.5 to $\sigma_1(e_{11})$, $\tau_2(f_{ij})$, $i = 1, 2, \ldots, k$; $\tau_2(f_{ik})$, $i = 2, \ldots, k$, we get a system of matrix units $\{F_{ij}\}_{1 \leq i, j \leq k}$ in $\mathcal{M}$ such that

1) $F_{11} + F_{22} + \ldots + F_{kk} = \sigma_1(e_{11})$ 
2) $\pi(F_{ij}) = \tau_2(f_{ij})$, $1 \leq i, j \leq k$

For each $j = 2, 3, \ldots, n$, $i = 2, 3, \ldots, k$, we put

3) $F_{i+j-1,k+1} = E_{ji} F_{11}$ and $F_{i+j-1,k+1} = E_{ji} F_{11}$

Then these partial isometries $\{F_{ij}\}_{2 \leq i \leq m}$ together with properties (1) and (2) furnish us with the desired $*$-monomorphism $\sigma_2 : \mathcal{A}_2 \to \mathcal{M}$.

§ 3. Liftings for UHF algebras

A $C^*$-algebra $\mathcal{A}$ with unit is uniformly hyperfinite (UHF) if there is an increasing sequence $\{\mathcal{A}_n\}$ of full matrix subalgebras containing the same unit of $\mathcal{A}$ and such that $\bigcup_{n=1}^{\infty} \mathcal{A}_n = \mathcal{A}$.

Theorem. Let $\mathcal{A}$ be a UHF algebra with $\bigcup_{n=1}^{\infty} \mathcal{A}_n = \mathcal{A}$, where $\mathcal{A}_n$ is increasing sequence of full matrix subalgebras. Let $\tau : \mathcal{A} \to \mathcal{M}(\mathbb{C})$ be a unital $*$-monomorphism. Then there exists a unital $*$-monomorphism $\sigma : \mathcal{A} \to \mathcal{M}$ such that $\pi \circ \sigma = \tau$.

Proof. Let $\sigma_1$ be a unital $*$-monomorphism of $\mathcal{A}_1$ into $\mathcal{M}$ such that $\pi \circ \sigma_1 = \tau |_{\mathcal{A}_1}$ (such a $\sigma_1$ exists by 2.6). By 2.11 we can extend $\sigma_1$ to a $*$-monomor-
phism $\sigma_2$ of $\mathcal{A}_2$ into $\mathcal{M}$ such that $\pi \circ \sigma = \tau | \mathcal{A}_2$. Thus by keeping doing this process, we get a unital $\ast$-monomorphism $\bar{\sigma} : \bigcup \mathcal{A}_n \to \mathcal{M}$ such that $\pi \circ \bar{\sigma} = \tau | \bigcup \mathcal{A}_n$. Let $\sigma$ be the unique extension of $\bar{\sigma}$ to $\mathcal{M}$. Now it is clear that $\pi \circ \sigma = \tau$. This completes the proof.

References


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