SCALE-INVARIANT MEASURABILITY
IN YEH-WIENER SPACE

BY KUN SOO CHANG

1. Introduction

Let $R = \{(s,t) : 0 \leq s \leq a, 0 \leq t \leq b\}$ and $C_2[R]$ be Yeh-Wiener space, i.e. $C_2[R] = \{x(\cdot, \cdot) : x(a,t) = x(s, \alpha) = 0, x(s, t) \text{ is continuous on } R\}$. $C_2[R]$ is often referred to as two parameter Wiener space. Let $a = s_0 < s_1 < \cdots < s_m = b$ and $a = t_0 < t_1 < \cdots < t_n = \beta$ and let $-\infty \leq a_{j,k} \leq b_{j,k} \leq +\infty$ be given for $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, n$. Let $E = (a_{11}, b_{11}) \times \cdots \times (a_{mn}, b_{mn}]$. $I = J(\alpha, \tau_1)(E) \equiv \{x \in C_2[R] : (x(s_1, t_1), \ldots, x(s_m, t_n)) \in E\}$ is called a strict interval of $C_2[R]$. If $E$ is an arbitrary measurable subset of $\mathbb{R}^{mn}$, then $I$ is called an interval of $C_2[R]$.

The collection $\mathcal{I}$ of all such strict intervals form a semi-algebra of subsets of $C_2[R]$. The measure of the strict interval $I$ is defined to be

$$m_1(I) = \int_{E} \omega(\overline{u} : \overline{s} : \overline{t}) \, d\overline{u},$$

where

$$\omega(\overline{u} : \overline{s} : \overline{t}) = \omega(u_{11}, \ldots, u_{mn}; s_1, \ldots, s_m; t_1, \ldots, t_n)$$

and $u_{0,k} = u_{j,0} = 0$ for all $j$ and $k$. This measure is countably additive on $\mathcal{I}$ and can be extended in the usual way to the $\sigma$-algebra $\sigma(\mathcal{I})$ generated by the strict intervals and then can be further extended so as to be a complete measure. This completed measure space is denoted by $(C_2[R], \mathcal{I}_1, m_1)$ and $\mathcal{I}_1$ is called the class of Yeh-Wiener measurable sets.

For $x \in C_2[R]$, let $\|x\| = \max_{(s, t) \in R} |x(s, t)|$. Then $(C_2[R], \|\cdot\|)$ is a separable Banach space.

Let $\mathcal{B}$ be the collection of all sets of the form $J(\alpha, \tau_1)(B)$ for all $(\overline{s} : \overline{t})$ and all Borel set $B$ in $\mathbb{R}^{mn}$. Then $\mathcal{B}$ is an algebra of subsets of $C_2[R]$. Let $\sigma(\mathcal{B})$ be the $\sigma$-algebra generated by $\mathcal{B}$ and $\mathcal{B}(C_2[R])$ be the class of Borel sets in $C_2[R]$. Then it is well known that $\sigma(\mathcal{B}) = \sigma(\mathcal{I}) = \mathcal{B}(C_2[R])$. $\sigma(\mathcal{I})$ is

Received May 20, 1982
sometimes referred to as the $\sigma$–algebra of strictly Yeh–Wiener measurable sets.

Let $\sigma_m$ be the partition:

$$\sigma_m = \{(s_j, t_k) : s_j = a + j\frac{(b-a)}{m}, t_k = a + k\frac{(b-a)}{m} : j, k = 1, 2, \ldots, m\}.$$

For each $x \in C_2[\mathbb{R}]$, let

$$S_{\sigma_m}(x) = \sum_{j=1}^{m} \sum_{k=1}^{m} \{x(s_j, t_k) - x(s_{j-1}, t_k) - x(s_j, t_{k-1}) + x(s_{j-1}, t_{k-1})\}^2.$$

For each $\lambda \geq 0$, let

$$C_\lambda = \{x \in C_2[\mathbb{R}] : \lim_{n \to \infty} S_{\sigma_n}(x) = \lambda^2 (b-a) (\beta - \alpha)/2\}$$

$$D = \{x \in C_2[\mathbb{R}] : \lim_{n \to \infty} S_{\sigma_n}(x) \text{ fails to exist}\}.$$

Note that $\nu C_2 = C_{\nu^2}$ for $\nu > 0$, $\lambda \geq 0$. Clearly $C_\lambda(\lambda \geq 0)$ and $D$ are Borel sets and $C_2[\mathbb{R}]$ is the disjoint union of this family of sets.

The key to our discussion is the following result due to Skoug [4].

**Theorem 1.1.** $m_1(C_1) = 1$.

In § 2 we will extend this result to partitions $\sigma_{h(n)}$ where $h$ is an increasing function from $\mathbb{N}$ into $\mathbb{N}$ such that $n \leq h(n)$ for all $n \in \mathbb{N}$.

**Definitions.** A set $E \subseteq C_2[\mathbb{R}]$ is said to be scale–invariant measurable if $\lambda E \subseteq \Upsilon_1$ for every $\lambda > 0$. A scale–invariant measurable set $N$ is called scale–invariant null if $m_1(\lambda N) = 0$ for every $\lambda > 0$. A property which holds except on a scale–invariant null set will be said to hold $s$–almost everywhere (denoted by $s$–a.e.).

In this paper we will extend the results on scale–invariant measurability in Wiener space which Johnson and Skoug obtained in [2] to Yeh–Wiener space. Many of the concepts, theorems and proofs will be much like analogous results in [2]. A number of the proofs will be omitted.

2. Preliminaries and Some Results in Yeh–Wiener Space

The following three propositions are well known results. We will state them without proof.

**Proposition 2.1.** $E$ is Lebesgue measurable in $\mathbb{R}^m$ iff $J_{(\mathcal{G}, \mathcal{F})}(E)$ is Yeh–Wiener measurable. In this case,

$$m_1(J_{(\mathcal{G}, \mathcal{F})}(E)) = \int_{\mathbb{R}^m} \omega(\mathbf{u} : \mathbf{s} : \mathbf{f}) d\mathbf{u}.$$
PROPOSITION 2.2. Let \( f(u_{11}, \ldots, u_{mn}) \) be a Lebesgue measurable function on \( \mathbb{R}^{mn} \) and \( F(x) = f(x(s_1, t_1), \ldots, x(s_m, t_n)) \). Then \( F \) is Yeh–Wiener measurable and
\[
\int_{C_2[\mathbb{R}]} F(x) \, dm_1(x) = \int_{\mathbb{R}^{mn}} f(u) \omega(u : \tilde{s} : \tilde{t}) \, du.
\]

Note that actually \( F(x) \) is Yeh–Wiener measurable iff \( f \) is Lebesgue measurable.

PROPOSITION 2.3. (a) If \( E \) is Yeh–Wiener measurable, then \( -E \) is Yeh–Wiener measurable and \( m_1E = m_1(-E). \)

(b) \[
\int_{C_2[\mathbb{R}]} F(x) \, dm_1(x) = \int_{C_2[\mathbb{R}]} F(-x) \, dm_1(x).
\]

Since \( \sigma(\mathcal{E}) = \mathcal{B}(C_2[\mathbb{R}]) \) we have that if \( E \) is a Borel set in \( \mathbb{R}^{mn} \), then \( J_{\tilde{x}, \tilde{t}}(E) \) is a Borel set in \( C_2[\mathbb{R}] \). The following proposition shows the converse to this fact. First of all we state a simple lemma.

LEMMA 2.4. Given any real numbers \( u_{ij} \), \( 0 \leq i \leq m, 0 \leq j \leq n \), let \( u \) denote the matrix \( (u_{ij}) \). Then there exists a piecewise linear continuous function \( H(u) \) on \( \mathbb{R} \) such that \( H(u)(s_i, t_j) = u_{ij} \); further, if \( u_{ij}^{(k)} \to u_{ij} \) as \( k \to \infty \) for \( 0 \leq i \leq m, 0 \leq j \leq n \), \( H(u^{(k)}) \to H(u) \) uniformly on \( \mathbb{R} \).

PROPOSITION 2.5. If \( J_{\tilde{x}, \tilde{t}}(E) \) is a Borel set in \( C_2[\mathbb{R}] \), then \( E \) is a Borel set in \( \mathbb{R}^{mn} \).

Proof. Define \( H \) on \( \mathbb{R}^{mn} \) as in Lemma 2.4 so that \( H(u)(s_i, t_j) = 0 \) if \( s = \alpha \) or \( t = \alpha \). Such an \( H \) is a continuous (and hence Borel) function from \( \mathbb{R}^{mn} \) to \( C_2[\mathbb{R}] \). Now \( X_E(u) = (X_{J_{\tilde{x}, \tilde{t}}(E)} \circ H)(u) \) since \( u \in E \) iff \( H(u) \in J_{\tilde{x}, \tilde{t}}(E) \). Suppose \( J_{\tilde{x}, \tilde{t}}(E) \) is a Borel set in \( C_2[\mathbb{R}] \). Then \( X_E = X_{J_{\tilde{x}, \tilde{t}}(E)} \circ H \) is a Borel function since it is the composition of two Borel functions. Hence \( E \) is a Borel subset of \( \mathbb{R}^{mn} \).

PROPOSITION 2.6. Let \( h : N \to N \) be an increasing function such that \( n \leq h(n) \) for all \( n \in N \). Let
\[
C_2^h = \{ x \in C_2[\mathbb{R}] : \lim_{t \to h(x)} S_{\alpha}(x) = \lambda (b-a)(\beta-\alpha)/2 \}.
\]

Then \( m_1(C_2^h) = 1 \).

Proof. Skoug [4, Proof of Lemma 1] showed that
\[
\int_{C_2[\mathbb{R}]} \left| S_{\delta h(x)}(x) - (b-a)(\beta-\alpha)/2 \right|^2 \, dx = \frac{1}{2} \left( (b-a)(\beta-\alpha)/h(n) \right)^2.
\]

Let \( E_n = \{ x : \left| S_{\delta h(x)}(x) - (b-a)(\beta-\alpha)/2 \right| \geq \frac{\log n}{\sqrt{2n}} (b-a)(\beta-\alpha) \}. \)
\[1/2 \left\{ \frac{(b-a)(\beta-\alpha)}{h(n)} \right\}^2 = \int_{C_2([R])} \left\{ S_{\frac{h(n)}{2}}(x) - \frac{(b-a)(\beta-\alpha)}{2} \right\}^2 dx \]
\[\geq \int_{E_n} \left\{ S_{\frac{h(n)}{2}}(x) - \frac{(b-a)(\beta-\alpha)}{2} \right\}^2 dx \]
\[\geq \frac{(\log n)^2}{2n} (b-a)^2 (\beta-\alpha)^2 \cdot m_1(E_n).\]

Hence \(m_1(E_n) \leq \frac{n}{[h(n) \log n]^2} \leq \frac{1}{n(\log n)^2} \)

Let \(F_n = \bigcup_{k=n}^\infty E_k\) and \(F = \bigcap_{n=1}^\infty F_n\). Then
\[m_1(F) \leq m_1(F_n) \leq \sum_{k=n}^\infty m_1(E_k) \leq \sum_{k=n}^\infty \frac{1}{k(\log k)^2} \to 0 \quad \text{as} \quad n \to \infty.\]

So \(m_1(F) = 0\). But for \(x \in F\), i.e. for \(x \in E_k\) for all \(k \geq n\) and for some \(n\),
\[\left| S_{\frac{h(n)}{2}}(x) - \frac{(b-a)(\beta-\alpha)}{2} \right| < \frac{\log k}{\sqrt{2k}} (b-a)(\beta-\alpha) \quad \text{for all} \quad k \geq n.\]

Hence \(\lim_{k \to \infty} S_{\frac{h(n)}{2}}(x) - \frac{(b-a)(\beta-\alpha)}{2} \leq \lim_{k \to \infty} \frac{\log k}{\sqrt{2k}} (b-a)(\beta-\alpha) = 0.\)

This implies that \(\lim_{k \to \infty} S_{\frac{h(n)}{2}}(x) = \frac{(b-a)(\beta-\alpha)}{2}\) for \(x \in F\). But \(m_1(F) = 0.\)

3. Scale-Invariant Measurable Sets in Yeh-Wiener Space

Let \(m_1\) be the Borel measure given by \(m_1(B) = m_1(\lambda^{-1}B)\) for \(B \in \mathcal{B}(C_2[R]).\)
Since \(\lambda^{-1}C_2 = C_1, m_1(C_2) = m_1(C_1) = 1\) by Theorem 1.1.

Let \(\mathcal{Y}_1\) denote the \(\sigma\)-algebra obtained by completing \((C_2[R], \mathcal{B}(C_2[R], m_2)\)
and let \(\mathcal{N}_1\) be the class of \(m_1\)-null sets. Note that every subset of \(C_2[R]\setminus C_1\) is in \(\mathcal{N}_1\). Let \(\mathcal{Y}\) and \(\mathcal{N}\) be the class of scale-invariant measurable sets and scale-invariant null sets, respectively.

**Proposition 3.1.**

(i) \(N\) is in \(\mathcal{N}_1\) iff \(\lambda^{-1}N\) is in \(\mathcal{N}_1\); equivalently, \(\mathcal{N}_1 = \lambda \mathcal{N}_1.\)
(ii) \(E\) is in \(\mathcal{Y}_1\) iff \(\lambda^{-1}E\) is in \(\mathcal{Y}_1\); equivalently, \(\mathcal{Y}_1 = \lambda \mathcal{Y}_1.\)
(iii) \(m_2(E) = m_1(\lambda^{-1}E)\) for \(E\) in \(\mathcal{Y}_1.\)

**Proof.**

(i) Let \(N\) be in \(\mathcal{N}_1\). Then \(N \subseteq M\) where \(M\) is an \(m_1\)-null Borel set. Hence \(m_1(\lambda^{-1}M) = m_1(M) = 0\) and so \(\lambda^{-1}M\) is an \(m_1\)-null Borel set. But then \(\lambda^{-1}N \subseteq \lambda^{-1}M\) is in \(\mathcal{N}_1\). The converse can be shown in essentially the same way.

(ii) Let \(E\) be in \(\mathcal{Y}_1\). Then \(E = B \cup N\) where \(B\) is in \(\mathcal{B}(C_2[R])\) and \(N\) is in \(\mathcal{N}_1\). Then \(\lambda^{-1}N\) is in \(\mathcal{N}_1\) by (i) and so \(\lambda^{-1}E = \lambda^{-1}B \cup \lambda^{-1}N\) is in \(\mathcal{Y}_1\). The rest of (ii) is easily checked.

(iii) Let \(E\) be in \(\mathcal{Y}_2\). Then \(E = B \cup M\) where \(B\) is in \(\mathcal{B}(C_2[R])\) and \(N\) is \(m_2\)-null. Then
\[m_2(E) = m_2(B \cup N) = m_2(B) = m_1(\lambda^{-1}B) = m_1(\lambda^{-1}B \cup \lambda^{-1}N) = m_1(\lambda^{-1}E).\]
PROPOSITION 3.2. \( \mathcal{Y} = \bigcap_{\lambda > 0} \mathcal{Y}_\lambda; \mathcal{Y} = \bigcup_{\lambda > 0} \mathcal{Y}_\lambda; \mathcal{Y} \) is a \( \sigma \)-algebra of subsets of \( C_2[\mathbb{R}] \).

REMARK. Beginning with this proposition, most of the proofs in the rest of this section are much like the proofs of corresponding results in [2]. We will include a few of these proofs but will omit most of them.

PROPOSITION 3.3. (i) \( E \) is in \( \mathcal{Y} \) iff \( E \cap C_2 \) is in \( \mathcal{Y}_\lambda \) for every \( \lambda > 0 \).

(ii) \( E \) is in \( \mathcal{N} \) iff \( E \cap C_2 \) is in \( \mathcal{N}_\lambda \) for every \( \lambda > 0 \).

The next theorem is quite simple. But it gives a very useful characterization of \( \mathcal{Y} \) and \( \mathcal{N} \) in that it shows rather well what scale-invariant measurable sets and scale-invariant null sets are really like and how they compare to Yeh–Wiener measurable sets and Yeh–Wiener null sets respectively.

THEOREM 3.4. (i) \( E \) is in \( \mathcal{Y} \) iff \( E \) has the form

\[
E = (\bigcup_{\lambda > 0} E_\lambda) \cup L,
\]

where each \( E_\lambda \) is an \( m_2 \)-measurable subset of \( C_2 \) and \( L \) is an arbitrary subset of \( C_0 \cup D \). Further, for \( E \) written in this manner, \( m_2(E) = m_2(E_\lambda) \) for all \( \lambda > 0 \).

(ii) \( N \) is in \( \mathcal{N} \) iff \( N \) has the form

\[
N = (\bigcup_{\lambda > 0} N_\lambda) \cup L,
\]

where each \( N_\lambda \) is an \( m_2 \)-null subset of \( C_2 \) and \( L \) is an arbitrary subset of \( C_0 \cup D \).

REMARK. The preceding theorem shows that there are many more Yeh–Wiener measurable sets than scale–invariant measurable sets: A set \( E \) is Yeh–Wiener measurable if and only if it has the form \( E_1 \cup L \) where \( E_1 \) is an \( m_1 \)-measurable subset of \( C_1 \) and \( L \) is an arbitrary subset of \( (\bigcup_{0 < i \leq 1} C_i) \cup D \cup C_0 \).

Similarly a set is Yeh–Wiener null if and only if it has the form \( N_1 \cup L \) where \( N_1 \) is an \( m_1 \)-null subset of \( C_1 \) and \( L \) is an arbitrary subset of \( (\bigcup_{0 < i \leq 1} C_i) \cup D \cup C_0 \).

Let \( a = s_0 < s_1 < \ldots < s_m = b, \alpha = t_0 < t_1 < \ldots < t_n = S \) and let \( E \) be any subset of \( \mathbb{R}^m \). Let

\[
Q = \bigcup_{\alpha \in \mathbb{Q}} \bigcup_{\lambda > 0} \bigcup_{(s_1, t_1) \in E} \bigcup_{(s_2, t_2) \in E} \ldots \bigcup_{(s_m, t_m) \in E} \{x \in C_2[\mathbb{R}] : (x(s_1, t_1), \ldots, x(s_m, t_m)) \in E\}.
\]

We have seen, in §2, that \( E \) is Borel measurable in \( \mathbb{R}^m \) if and only if \( Q \) is Borel measurable in \( C_2[\mathbb{R}] \) and that \( E \) is Lebesgue measurable in \( \mathbb{R}^m \) if and only if \( Q \) is Yeh–Wiener measurable [3]. It is easy to see that such sets \( Q \) are scale–invariant measurable, since for any \( \lambda > 0 \),

\[
\lambda Q = \{x \in C_2[\mathbb{R}] : (x(s_1, t_1), \ldots, x(s_m, t_m)) \in \lambda^{-1}E\}
\]
is Yeh–Wiener measurable.

**Proposition 3.5.** For every \( \lambda > 0 \), \( B(C_2^2[R]) \subset \mathcal{I} \subset \mathcal{J}_0 \).

The following result of Skoug [4] becomes rather transparent using Theorem 3.4.

**Corollary 3.6.** Let \( f \) be any function with domain \( (0, \infty) \) and satisfying \( 0 \leq f(\lambda) \leq 1 \). Then there exists \( E \) in \( \mathcal{I} \) such that \( m_1(\lambda E) = f(\lambda) \) for all \( \lambda > 0 \).

**Proof.** For each \( \lambda > 0 \), pick \( E \in C_1 \) such that \( E \) is in \( \mathcal{I} \) and \( m_1(E) = f(\lambda^{-1}) \). (Such \( E \) exists by the following lemma.) Then \( E = \bigcup_{n=1}^{\infty} E_n \) is the desired set since, by Proposition 3.1 and Theorem 3.4, we have \( m_1(\lambda E) = m_1(E) = m_1(E) \).

**Lemma.** Given \( \gamma \in [0,1] \), there exists \( E \in C_1 \) such that \( E \in \mathcal{I} \) and \( m_1(E) = \gamma \) for each \( \lambda > 0 \).

**Proof.** Given \( \gamma \in [0,1] \), there exists a real number \( a_\gamma \) such that

\[
\frac{1}{\sqrt{\pi (b-a) (\beta - \alpha)}} \int e^{-\frac{u^2}{(b-a) (\beta - \alpha)}} du = \gamma.
\]

Let \( E = \{ x \in C_2^2[R] : -\infty < x(b, \beta) \leq a_\gamma \} \). Then \( E \) is in \( \mathcal{I} \) and \( m_1(E) = \gamma \). Let \( E_\gamma = E \cap C_1 \). Then \( E_\gamma \in C_1 \) and \( m_1(E_\gamma) = m_1(E) = \gamma \). Let \( E_\lambda = \lambda E_\gamma \). Then \( E_\lambda \) is in \( \mathcal{I} \) and \( E_\lambda \in \mathcal{I} \).

Our sets \( C_1, \lambda \geq 0 \) and \( D \) depend on the particular sequence of partitions on \( R \) that we choose. If \( \sigma_{h,(a)} \) denotes another sequence of partitions, we may let

\[
C_1^{h,(a)} = \{ x \in C_2^2[R] : \lim_{n \to \infty} S_{h,(a)}^n(x) = \lambda^2 (b-a)(\beta - \alpha)/2 \}
\]

and

\[
D^{h,(a)} = \{ x \in C_2^2[R] : \lim_{n \to \infty} S_{h,(a)}^n(x) \text{ fails to exist} \}.
\]

Essentially because of Proposition 2.6, all of the results obtained up to this point, with changes in notation where appropriate, go through. Note, however, that \( \mathcal{I}, \mathcal{J}, m_1, \mathcal{N} \) and \( \% \) are all independent of the sequence of partitions. A set \( E \) in \( \mathcal{N} \) now has two decompositions according to the two versions of Theorem 3.4:

\[
E = (\bigcup_{n=0}^{\infty} E_n) \cup L = (\bigcup_{n=0}^{\infty} E_n^h) \cup L^h
\]

where \( E_n^h = E \cap C_n^h \) and \( L^h = E \cap (C_0^h \cup D^h) \). How do these two decompositions relate to one another? The next proposition shows that they agree up to a scale–invariant null set.

**Proposition 3.7.** The two decompositions of \( E \) given by (3.4) have the property that the set

\[
E = (\bigcup_{n=0}^{\infty} E_n) \cup L = (\bigcup_{n=0}^{\infty} E_n^h) \cup L^h
\]
Scale-invariant measurability in Yeh–Wiener space

\[(3.5) \quad (\bigcup_{\lambda > 0} E_\lambda \Delta E_\lambda) \cup (L \Delta L^k) \]

is scale-invariant null.

**Proof.** First note that for all \( \lambda > 0 \)
\[
m_2(E_\lambda \setminus E_\lambda^k) = m_2[(E \cap C_\lambda) \setminus (E \cap C_\lambda^k)]
\]
\[
= m_2(E \cap (C_\lambda \setminus C_\lambda^k))
\]
\[
\leq m_2(C_\lambda \setminus C_\lambda^k)
\]
\[
\leq m_1(C_\lambda \setminus [R \setminus C_\lambda]) = 0.
\]

Thus by Theorem 3.4, the set \( \bigcup_{\lambda > 0} (E_\lambda \setminus E_\lambda^k) \cup (L \setminus L^k) \) is scale-invariant null.

In similar fashion one can show that the set \( \bigcup_{\lambda > 0} (E_\lambda^k \setminus E_\lambda) \cup (L^k \setminus L) \) is scale-invariant null which concludes the proof since
\[
\bigcup_{\lambda > 0} (E_\lambda \Delta E_\lambda) \cup (L \Delta L^k) = \bigcup_{\lambda > 0} (E_\lambda \setminus E_\lambda^k) \cup (L \setminus L^k) \cup \{ \bigcup_{\lambda > 0} (E_\lambda^k \setminus E_\lambda) \cup (L^k \setminus L) \}.
\]

This paper is based on Chapter 2 of the author’s Ph. D. Thesis [1] written at the University of Nebraska under the direction of Professor Gerald W. Johnson.

**References**


Yonsei University