ON THE LOCAL CONVERGENCE IN MEASURE

BY DONG M. CHUNG

Let $(X, \mathcal{B}, \mu)$ be a measure space. Let $\mathcal{M}$ denote the set of all real valued measurable functions on $X$, where we have identified functions which are equal $\mu$-almost everywhere. It is well known that if $\mu$ is finite, $\mathcal{M}$ is a complete metric space with the metric

$$d(f, g) = \int \frac{|f - g|}{1 + |f - g|} d\mu, \quad f, g \in \mathcal{M}$$

which is compatible with convergence in measure. Indeed, this fact follows from the following known result [1].

**THEOREM 1.** Let $\langle f_n \rangle$ be a sequence of real valued measurable functions on $(X, \mathcal{B}, \mu)$ which is Cauchy in measure. Then there exists a subsequence which converges almost everywhere and in measure to a real valued measurable function $f$ on $X$.

In this note we shall show that if $\mu$ is $\sigma$-finite, Theorem 1 can be extended (see Theorem 2) to apply to sequences of real valued measurable functions which are Cauchy in weaker type of convergence than convergence in measure. Now we begin with a definition.

**DEFINITION.** Let $f, f_1, f_2, \ldots$ be real valued measurable functions on $(X, \mathcal{B}, \mu)$. The sequence $\langle f_n \rangle$ is said to converge locally in measure to $f$ if for each $\alpha > 0$, and $E \in \mathcal{B}$ with $\mu(E) < \infty$

$$\lim_{m \to \infty} \mu(\{ x \in E : |f_n(x) - f(x)| \geq \alpha \}) = 0.$$

The sequence $\langle f_n \rangle$ is said to be Cauchy locally in measure if for each $\alpha > 0$, and $E \in \mathcal{B}$ with $\mu(E) < \infty$

$$\lim_{m, n \to \infty} \mu(\{ x \in E : |f_m(x) - f_n(x)| \geq \alpha \}) = 0.$$

Obviously, if $\langle f_n \rangle$ converges in measure to $f$, then $\langle f_n \rangle$ converges locally in measure to $f$, but the converse need not be valid. For example, with $X = [0, \infty]$ and Lebesgue measure $\mu$, the sequence

$$f_n(x) = \begin{cases} 1, & n < x \leq n + 1 \\ 0, & \text{otherwise} \end{cases}$$

satisfies $\lim_{n \to \infty} f_n(x) = 0$ for all $x$, but $\langle f_n \rangle$ is not Cauchy locally in measure.
converges to zero locally in measure but fails to converge to zero in measure. Thus local convergence in measure is weaker than convergence in measure.

**Theorem 2.** Let \((X, \mathcal{B}, \mu)\) be a \(\sigma\)-finite measure space and \(<f_n>\) a sequence of real valued measurable functions on \(X\) which is Cauchy locally in measure. Then there exists a subsequence which converges almost everywhere and locally in measure to a real valued measurable function \(f\) on \(X\).

**Proof.** Since \(\mu\) is \(\sigma\)-finite, there exists a sequence \(<E_n>\) of disjoint subsets of \(X\) in \(\mathcal{B}\) such that \(\mu(E_n) < \infty\) and \(\bigcup_{n=1}^{\infty} E_n = X\). Denote by \(\mu_n\) the restriction of \(\mu\) to \(E_n\), by \(\mathcal{B}(E_n)\) the \(\sigma\)-algebra \(\{B \cap E_n : B \in \mathcal{B}\}\), and by \(f_n^{(m)}\) the restriction of \(f_n\) to \(E_n\). Since the sequence \(<f_n^{(1)}>\) is a Cauchy sequence of real valued measurable functions on \((E_1, \mathcal{B}(E_1), \mu_1)\), it follows from Theorem 1 that there exists a subsequence \(<f_{1,k}^{(1)}>\) of \(<f_n^{(1)}>\) which converges \(\mu_1\)-almost everywhere. Thus we have \(N_1 \in \mathcal{B}(E_1)\) such that \(\mu_1(N_1) = 0\) and \(<f_{1,k}^{(1)}(x)>\) converges for each \(x \in E_1 - N_1\). Since the sequence \(<f_{1,k}>\) is a subsequence of \(<f_n>\), by the same argument as above, we can choose a subsequence \(<f_{2,k}>\) of \(<f_{1,k}>\) such that \(<f_{2,k}^{(2)}(x)>\) converges \(\mu_2\)-almost everywhere. Thus we have \(N_2 \in \mathcal{B}(E_2)\) such that \(\mu_2(N_2) = 0\) and \(<f_{2,k}^{(2)}(x)>\) converges for each \(x \in E_2 - N_2\). If we continue this process, for each \(j \geq 1\), we will obtain a subsequence \(<f_{j,k}>\) of \(<f_n>\) such that \(<f_{j,k}^{(j)}(x)>\) converges on \((E_j, \mathcal{B}(E_j), \mu_j)\) off a \(\mu_j\) null set \(N_j\). Furthermore, \(<f_{j,k}>\) is a subsequence of \(<f_{j-1,k}>\), where \(f_{0,k} = f_k\) for all \(k\). Now let \(N = \bigcup_{j=1}^{\infty} N_j\), then \(\mu(N) = \sum_{j=1}^{\infty} \mu_j(N_j) = 0\) since \(E_j\)'s are disjoint. Hence the subsequence \(<f_{h,k}>\) of \(<f_n>\) converges on \(X\) off the \(\mu\) null set \(N\). If we define \(f\) by

\[
f(x) = \begin{cases} 
  f_{h,k}(x), & x \in N \\
  0, & x \notin N 
\end{cases}
\]

then the subsequence \(<f_{h,k}>\) converges almost everywhere to the real valued measurable function \(f\) on \(X\).

To see that \(<f_{h,k}>\) converges locally in measure to \(f\), for each \(n\) let \(f^{(n)}\) denote the restriction of \(f\) to \(E_n\). Since \(<f_{h,k}^{(n)}(x)>\) converges to \(f^{(n)}\) almost everywhere on the finite measure space \((E_n, \mathcal{B}(E_n), \mu_n)\) it follows from Egoroff's theorem ([1], p.74) that \(<f_{h,k}^{(n)}(x)>\) converges to \(f^{(n)}\) in measure on \(E_n\). Hence \(<f_{h,k}>\) converges locally in measure to \(f\).

**Corollary 3.** Let \((X, \mathcal{B}, \mu)\) and \(<f_n>\) be as in Theorem 2. Then there...
exists a real valued measurable function on \( X \) to which the sequence converges locally in measure. This limit function \( f \) is uniquely determined almost everywhere.

Proof. We have seen that there exists a subsequence \( \langle f_{k_{i}} \rangle \) of \( \langle f_n \rangle \) which converges locally in measure to a function \( f \). Now it is enough to show that the original sequence converges locally in measure to \( f \). However, this can be done by retracing the proof of Corollary 7.7 [1] with \( E_n \) in place of \( X \), where \( E_n \) is as in the proof of Theorem 2.

Let \( (X, \mathcal{B}, \mu) \) be a \( \sigma \)-finite measure space and \( \langle B_n \rangle \) an increasing sequence of subsets of \( X \) in \( \mathcal{B} \) with \( \mu(B_n) < \infty \). Define a metric \( d_n \) on \( M \) by

\[
d_n(f, g) = \int_{B_n} \frac{|f-g|}{1+|f-g|} d\mu, \quad f, g \in M.
\]

Then, it is easy to see that local convergence in measure is compatible with the metric on \( M 
\]

\[
d(f, g) = \sum_{n=1}^{\infty} 2^{-n} d_n(f, g).
\]

Thus, Corollary 3 shows that \( m \) becomes a complete metric space with the metric \( d \).

The following corollary can be easily shown by using Theorem 2 and standard arguments (see [1]).

**COROLLARY 4.** Let \( (X, \mathcal{B}, \mu) \) be a \( \sigma \)-finite measure space. Then Fatou’s Lemma and the Monotone and Lebesgue Dominated Convergence theorem on \( X \) remain valid if convergence in measure is replaced by local convergence in measure.

**Reference**