THE $T_\theta$-TOPOLOGY AND FAINTLY CONTINUOUS FUNCTIONS

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1. Introduction

For a topological space $X$ and $A \subseteq X$, the $\theta$-closure of $A$ is defined [9] to be the set of all $x \in X$ such that every closed neighborhood of $x$ intersects $A$ non-emptily and is denoted by $\text{Cl}_\theta(A)$. The subset $A$ is called $\theta$-closed if $\text{Cl}_\theta(A) = A$. In a similar manner, the $\theta$-interior of a set $A \subseteq X$ is defined to be the set of all $x \in A$ for which there exists a closed neighborhood of $x$ contained in $A$. The $\theta$-interior of $A$ is denoted by $\text{Int}_\theta(A)$. In particular, the concept of $\theta$-closed sets has been extensively studied by Professors Velicko [9], Dickman and Porter [1], Joseph [3] and others. With the definition of the $\theta$-interior of a set, a new topology will be described which is related to the semi-regular topology on $(X, T)$. The semi-regular topology, denoted by $T_s$, is the topology having as its base the set of all regular-open sets in $(X, T)$ [2, Problem 22, p.92]. Recall that a set $A$ is regular-open provided $\text{Int}(\text{Cl}(A)) = A$. Specifically, for any set $A$, $\text{Int}(\text{Cl}(A))$ is always regular-open.

2. The $T_\theta$-topology

DEFINITION 1. An open set $U$ in $(X, T)$ is called $\theta$-open if $\text{Int}_\theta(U) = U$.

From the definition of $\theta$-closed sets, it follows that the complement of a $\theta$-open set is $\theta$-closed and the complement of a $\theta$-closed set is $\theta$-open. According to [9], the intersection of $\theta$-closed sets is $\theta$-closed and the finite union of $\theta$-closed sets is a $\theta$-closed set. Therefore, arbitrary unions and finite intersections of $\theta$-open sets are themselves $\theta$-open. Consequently, the collection of $\theta$-open sets in a topological space $(X, T)$ form a topology $T_\theta$ on $X$ which we call the $T_\theta$-topology. Evidently, $T = T_\theta$ if and only if $(X, T)$ is regular.

THEOREM 1. Let $X$ be any topological space. If $V \subseteq X$ is $\theta$-open and $x \in V$, then there exists a regular-open set $U$ such that $x \in U \subseteq \text{Cl}(U) \subseteq V$.

PROOF. Since $V$ is $\theta$-open, there exists an open set $W$ such that $x \in W \subseteq \text{Cl}(W) \subseteq V$. But $\text{Int}(\text{Cl}(W)) = U$ is regular-open and it follows that $x \in U \subseteq \text{Cl}(U) \subseteq V$ due to the fact that $\text{Cl}(W) \subseteq V$. 
COROLLARY TO THEOREM 1. The set $V$ is $\theta$-open if and only if for each $x \in V$ there exists a regular-open $U$ such that $x \in U \subseteq \text{Cl}(U) \subseteq V$.

Theorem 1 implies that in any topological space, $T_\theta \subseteq T_s$. The converse need not be true as the next example shows.

EXAMPLE 1. The topologies $T_s$ and $T_\theta$ may be different even in a completely Hausdorff space. Let $X = (0, 2)$ be a subset of the reals $R$ with the usual topology. For each $k \in \mathbb{N}$, define $H_k = \bigcup \left( \left( -\frac{2n+1}{2n(n+1)}, \frac{2n-1}{2n(n-1)} \right) \right)$ for $n > k$, $n$ even, and topologize $X$ using the following subbasic open sets: $\{ V \subseteq X - \{1\} : V \text{ open in } R \} \cup \{ H_k \cup G : k \in \mathbb{N}, G \subseteq X, G \text{ open in } R \text{ and contains the point 1} \}$. Then $U = (3/4, 3/2) \cup H_1$ is regular-open, but not $\theta$-open. Consequently, $T_s \neq T_\theta$.

THEOREM 2. Let $A \subseteq X$ be $\theta$-closed and let $x \in A$. Then there exists regular-open sets which separate $x$ and $A$.

PROOF. Since $X - A$ is $\theta$-open and contains $x$, there exists a regular-open set $U$ such that $x \in U \subseteq \text{Cl}(U) \subseteq V$ by Theorem 1. Now $\text{Int}(\text{Cl}(X - \text{Cl}(U)))$ is nonempty, regular-open, contains $A$ and is disjoint from $U$.

A space is defined to be almost-regular [8] if for each $x \in X$ and regular-closed $A$ not containing $x$, there exist disjoint open sets $U$ and $V$ such that $x \in U$ and $A \subseteq V$.

THEOREM 3. Let $X$ be almost-regular. Then each regular-open set in $X$ is also $\theta$-open.

PROOF. Since $X$ is almost-regular, for each regular-open $V$ in $X$ and $x \in V$ there exists a regular-open $U$ such that $x \in U \subseteq \text{Cl}(U) \subseteq V$ according to Theorem 2.2 of [8]. Thus each point of $V$ has a closed neighborhood contained in $V$ implying that $V$ is $\theta$-open.

COROLLARY TO THEOREM 3. If $(X, T)$ is almost-regular, then $T_s = T_\theta$.

PROOF. By Theorem 3, $T_s \subseteq T_\theta$ and by Theorem 1, $T_\theta \subseteq T_s$. Therefore, $T_s = T_\theta$.

THEOREM 4. The space $(X, T)$ is almost-regular if and only if $T_s = T_\theta$.

PROOF. If $(X, T)$ is almost-regular, then $T_s = T_\theta$ by the Corollary to Theorem 3. Conversely, if $T_s = T_\theta$, let $V$ be a regular-open set in $(X, T)$ and let $x \in V$. Then $V$ is also $\theta$-open and by Theorem 1 there exists a regular-open set $U$ such that $x \in U \subseteq \text{Cl}(U) \subseteq V$. Consequently, $(X, T)$ is almost-regular by Theorem 2.2.
The Topology and Faintly Continuous Functions

THEOREM 5. Let \( X \) and \( Y \) be topological spaces. If \( U \subseteq X \) and \( V \subseteq Y \) are \( \theta \)-open, then \( U \times V \) is \( \theta \)-open in \( X \times Y \).

PROOF. Let \((x, y) \in U \times V\). Then there exist open sets \( U_1 \) and \( V_1 \) such that \( x \in U_1 \subseteq \text{Cl}(U_1) \subseteq U \) and \( y \in V_1 \subseteq \text{Cl}(V_1) \subseteq V \) because both \( U \) and \( V \) are \( \theta \)-open. Therefore, \((x, y) \in \text{Cl}(U_1) \times \text{Cl}(V_1) = \text{Cl}(U_1 \times V_1) \subseteq U \times V\). Consequently, each point of \( U \times V \) has a closed neighborhood contained in \( U \times V \) which shows \( U \times V \) is \( \theta \)-open.

THEOREM 6. Let \( W \) be \( \theta \)-open in the product space \( \prod_{\alpha \in J} X_{\alpha} \). Then each projection \( \Pi_{\alpha}(W) \) is \( \theta \)-open in \( X_{\alpha} \).

PROOF. Let \( y_{\alpha} \in \Pi_{\alpha}(W) \) and let \( \{y_{\alpha}\} \) be a point in \( W \) such that \( \Pi_{\alpha}(y_{\alpha}) = y_{\alpha} \).

Now since \( W \) is \( \theta \)-open, there exists a basic open set \( U = U_{\alpha_1} \times U_{\alpha_2} \times \ldots \times U_{\alpha_n} \times \prod_{\alpha \neq \alpha_1, \ldots, \alpha_n} X_{\alpha} \) such that \( \{y_{\alpha}\} \in U \subseteq \text{Cl}(U) = \text{Cl}(U_{\alpha_1}) \times \text{Cl}(U_{\alpha_2}) \times \ldots \times \text{Cl}(U_{\alpha_n}) \times \prod_{\alpha \neq \alpha_1, \ldots, \alpha_n} X_{\alpha} \subseteq W \).

Without loss of generality, we may assume that for some \( 1 \leq j \leq n, \alpha = \alpha_j \). Thus, \( y_{\alpha} \in \Pi_{\alpha_1}(U_{\alpha_1}) \subseteq \Pi_{\alpha_1}(W) \) so that each point of \( \Pi_{\alpha_1}(W) \) contains a closed neighborhood lying in \( \Pi_{\alpha_1}(W) \). It follows that \( \Pi_{\alpha_1}(W) \) is \( \theta \)-open.

THEOREM 7. Let \( f : X \to Y \) be a function from \( X \) onto \( Y \) that is both open and closed. Then \( f \) preserves \( \theta \)-open sets.

PROOF. Let \( U \) be \( \theta \)-open in \( X \) and let \( y \in f(U) \). Then there exists an \( x \in U \) such that \( f(x) = y \) and an open set \( U_0 \) such that \( x \in U_0 \subseteq \text{Cl}(U_0) \subseteq U \). Therefore, \( f(x) = y \in f(U_0) \subseteq f(\text{Cl}(U_0)) \subseteq f(U) \). Now, the fact that \( f \) is both open and closed shows that \( f(U_0) \) is an open set whose closure \( \text{Cl}(f(U_0)) \subseteq f(\text{Cl}(U_0)) = f(\text{Cl}(U_0)) \) is contained in \( f(U) \). This shows that \( f(U) \) is \( \theta \)-open.

THEOREM 8. Let \( f : X \to Y \) be continuous. If \( V \subseteq Y \) is \( \theta \)-open, then \( f^{-1}(V) \) is \( \theta \)-open in \( X \).

PROOF. Let \( x \in f^{-1}(V) \). Then \( f(x) \in V \) and there exists an open set \( U \) such that \( f(x) \in U \subseteq \text{Cl}(U) \subseteq V \) because \( V \) is \( \theta \)-open. Thus, \( x \in f^{-1}(U) \subseteq f^{-1}(\text{Cl}(U)) \subseteq f^{-1}(V) \). The continuity of \( f \) then gives \( f^{-1}(U) \) as an open set whose closure is contained in \( f^{-1}(V) \) which shows that \( f^{-1}(V) \) is \( \theta \)-open.

3. Faintly-continuous functions

DEFINITION 2. Let \( X \) and \( Y \) be topological spaces. Then \( f : X \to Y \) is faintly-continuous if for each \( x \in X \) and \( \theta \)-open \( V \) containing \( f(x) \), there exists an open
set $U$ containing $x$ such that $f(U) \subseteq V$.

As will be demonstrated shortly, the concept of faintly-continuous is a very weak form of continuity. Perhaps the concept could have been better named \(\theta\)-continuous, but that notation is already reserved for a different kind of non-continuous function. (See, for example, Definition 2 of [6].)

**THEOREM 9.** Let $f : X \to Y$ be given. Then they are equivalent:
(a) $f : X \to (Y, T)$ is faintly-continuous.
(b) $f : X \to (Y, T_0)$ is continuous.
(c) The inverse image of each \(\theta\)-open set in $(Y, T)$ is open in $X$.
(d) The inverse image of each \(\theta\)-closed set in $(Y, T)$ is closed in $X$.

**PROOF.** The implications follow easily from the definitions.

A function $f : X \to Y$ is called weakly-continuous [4] if for each $x \in X$ and each open set $V$ containing $f(x)$ there exists an open set $U$ containing $x$ such that $f(U) \subseteq \text{Cl}(V)$.

**THEOREM 10.** If $f : X \to Y$ is weakly-continuous, then $f$ is faintly-continuous.

**PROOF.** Let $x \in X$ and let $V$ be a \(\theta\)-open set containing $f(x)$. Then there exists an open set $W$ such that $f(x) \in W \subseteq \text{Cl}(W) \subseteq V$. Now, since $f$ is weakly-continuous, there exists an open set $U$ containing $x$ such that $f(U) \subseteq \text{Cl}(W) \subseteq V$. Consequently $f$ is faintly-continuous.

**EXAMPLE 2.** A faintly-continuous function need not be weakly-continuous. Let $X = [0, 1]$ with topology $\{\emptyset, X, \{1\}\}$ and let $Y = \{a, b, c\}$ with topology $\{\emptyset, Y, \{a\}, \{b\}, \{a, b\}\}$. Finally, let $f : X \to Y$ be defined as $f(0) = a$ and $f(1) = b$. Then $f$ is not weakly-continuous at $x=0$, but $f$ is faintly-continuous since the only \(\theta\)-open set in $Y$ is $Y$ itself.

Theorem 10 and Example 2 now allow us to see the position faintly-continuous functions occupy among other well-known non-continuous functions. First, however, we should recall the definitions of almost-continuity and \(\theta\)-continuity:
a function $f : X \to Y$ is almost-continuous (\(\theta\)-continuous) if for each $x \in X$ and each regular-open $V$ (open $V$) containing $f(x)$, there exists an open $U$ containing $x$ such that $f(U) \subseteq V$ ($f(\text{Cl}(U)) \subseteq \text{Cl}(V)$). Now it readily follows that

continuity$\Rightarrow$almost-continuity$\Rightarrow$\(\theta\)-continuity$\Rightarrow$weak-continuity$\Rightarrow$faint-continuity.

These implications, aside from the last one, are explored in [6].

**THEOREM 11.** Let $(Y, T)$ be an almost-regular space and $f : X \to (Y, T)$ a
faintly-continuous function. Then \( f \) is almost-continuous.

**PROOF.** Since \( f : X \rightarrow (Y, T) \) is faintly-continuous, then \( f : X \rightarrow (Y, T_0) \) is continuous. But \((Y, T)\) almost-regular implies \(T_0 = T_3\) by the Corollary to Theorem 3. Thus, \( f : X \rightarrow (Y, T_3) \) is continuous showing that \( f : X \rightarrow (Y, T) \) is almost-continuous.

**COROLLARY TO THEOREM 11.** If \((Y, T)\) is almost-regular and \( f : Y \rightarrow (Y, T) \), then they are equivalent:

(a) \( f \) is faintly-continuous.
(b) \( f \) is weakly-continuous.
(c) \( f \) is \( \theta \)-continuous.
(d) \( f \) is almost-continuous.

In the above Corollary, if almost-regular is replaced with regular, then we may add continuity to the list of equivalences.

**THEOREM 12.** If \( f : X \rightarrow Y \) is faintly-continuous and \( A \subseteq X \), then \( f \mid A : A \rightarrow Y \) is faintly-continuous.

**PROOF.** Evident.

For a given \( f : X \rightarrow Y \), the graph map \( g : X \rightarrow X \times Y \) is defined as \( g(x) = (x, f(x)) \).

**THEOREM 13.** If the graph map of \( f : X \rightarrow Y \) is faintly-continuous, then \( f \) is faintly-continuous.

**PROOF.** Let \( x \in X \) and let \( V \) be \( \theta \)-open in \( Y \) containing \( f(x) \). Then \( X \times V \) is \( \theta \)-open in \( X \times Y \) by Theorem 5 and contains \( g(x) = (x, f(x)) \). Since the graph map \( g : X \rightarrow X \times Y \) is faintly-continuous, there exists an open set \( U \) containing \( x \) such that \( g(U) \subseteq X \times V \). This implies that \( f(U) \subseteq V \) so that \( f \) is faintly-continuous.

**THEOREM 14.** If \( f : X \rightarrow Y \) is weakly-continuous, then the graph map \( g : X \rightarrow X \times Y \) is faintly-continuous.

**PROOF.** Let \( x \in X \) and let \( W \) be a \( \theta \)-open set containing \( g(x) \). Then there is a closed neighborhood, hence a closed basic open set \( \text{Cl}(U \times V) \), containing \( g(x) \) and lying inside \( W \). Thus, \( g(x) = (x, f(x)) \subseteq \text{Cl}(U \times V) = \text{Cl}(U) \times \text{Cl}(V) \) so that \( f(x) \subseteq \text{Cl}(V) \). Since \( f \) is weakly-continuous, there exists an open set \( U_0 \subseteq U \) containing \( x \) such that \( f(U_0) \subseteq \text{Cl}(V) \). Consequently, \( g(U_0) \subseteq \text{Cl}(U) \times \text{Cl}(V) \subseteq W \) show-
ing $g$ to be faintly-continuous.

3. Functions with extremely-closed graphs

**DEFINITION 3.** The graph $G(f)$ of $f : X \to Y$ is **extremely-closed** if for each $(x, y) \in G(f)$ there exists an open set $U$ containing $x$ and a $\theta$-open set $V$ containing $y$ such that $(U \times V) \cap G(f) = \emptyset$.

The proofs to the next two theorems follow easily from the above definition.

**THEOREM 15.** The graph of $f : X \to Y$ is extremely-closed if and only if for each $x \in X$ and $y \neq f(x)$ there exists an open set $U$ containing $x$ and a $\theta$-open set $V$ containing $y$ such that $f(U) \cap V = \emptyset$.

**THEOREM 16.** The graph of $f : X \to (Y, T)$ is extremely-closed if and only if the graph of $f : X \to (Y, T_\emptyset)$ is closed.

**THEOREM 17.** If $f : X \to (Y, T)$ is faintly-continuous and $(Y, T_\emptyset)$ is Hausdorff, then $f$ has an extremely-closed graph.

**PROOF.** We know that $f : X \to (Y, T_\emptyset)$ is continuous because $f : X \to (Y, T)$ is faintly-continuous. Since $T_\emptyset$ is Hausdorff, the graph of $f : X \to (Y, T_\emptyset)$ is closed [2, Theorem 1, 5(3), p.140]. Thus, $f : X \to (Y, T)$ has an extremely-closed graph by Theorem 16.

**THEOREM 18.** Let $Y$ be completely Hausdorff and let $f : X \to Y$ be faintly-continuous. Then $G(f)$ is extremely-closed.

**PROOF.** Let $x \in X$ and let $y \neq f(x)$. Since $Y$ is completely Hausdorff, there exists a continuous $g : Y \to R$ such that $g(f(x)) \neq g(y)$. Thus, there exist open disjoint sets $W$ and $G$ containing $g(f(x))$ and $g(y)$, respectively, such that $g^{-1}(W) \cap g^{-1}(G) = \emptyset$. But $g^{-1}(W)$ is $\theta$-open by Theorem 8 and the fact that every open subset of $R$ is $\theta$-open. Therefore, there exists an open $U$ containing $x$ such that $f(U) \subseteq g^{-1}(W)$ so that $f(U) \cap g^{-1}(G) = \emptyset$. Theorem 15 now implies that the graph of $f$ is extremely-closed.

The graph of $f : X \to Y$ is called **strongly-closed** [5] if for each $(x, y) \in G(f)$ there exist open sets $U$ and $V$ containing $x$ and $y$, respectively, such that $(U \times \text{Cl}(V)) \cap G(f) = \emptyset$.

**THEOREM 19.** Let $f : X \to Y$ have an extremely-closed graph. Then $f$ has a strongly-closed graph.
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PROOF. Let \( x \in X \) and \( y \neq f(x) \). Then by Theorem 15, there exists an open set \( U \) containing \( x \) and a \( \theta \)-open set \( V \) containing \( y \) such that \( f(U) \cap V = \emptyset \). Since \( V \) is \( \theta \)-open, there exists an open set \( V_0 \) such that \( y \in V_0 \subseteq \text{Cl}(V_0) \subseteq V \) so that \( f(U) \cap \text{Cl}(V_0) = \emptyset \). It follows that the graph of \( f \) is strongly-closed by the first Lemma of [7].

From Theorem 19 and [5] we now see the position of extremely-closed graphs as follows:

extremely-closed graph \( \Rightarrow \) strongly-closed graph \( \Rightarrow \) closed graph. It is shown in [5] that a closed graph need not be strongly-closed. Our last example shows the first implication above cannot, in general, be reversed.

EXAMPLE 3. Let \( Y = \{0, 2\} \) and let \( G_k \) be defined by

\[
G_k = \bigcup \left\{ \left[ \frac{2n+1}{2n(n+1)}, \frac{2n-1}{2n(n-1)} \right) : n > k, \ n \text{ is odd} \right\}, \ k \in \mathbb{N}.
\]

Let \( H_k \) be defined as in Example 1 and topologize \( Y \) using the following subbasic open sets: \( \{ V \subseteq Y - \{1\} : V \text{ open in } R \} \cup \{ H_k \cup G : k \in \mathbb{N}, \ G \subseteq Y, \ G \text{ open in } R \) and contains the point 1 \} \cup \{ G_k \cup 0 : k \in \mathbb{N} \}. \) Now define \( f : X \rightarrow Y \) by \( f(x) = x \) for all \( x \in X \) where \( X \) is the space given in Example 1. Then \( f \) is continuous and \( Y \) is Hausdorff which implies \( G(f) \) is strongly-closed by the Corollary to Theorem 1 of [5]. However, the point \( (1, 0) \in G(f) \), but for each open \( U \) containing 1 and each \( \theta \)-open set \( V \) containing 0, \( (U \times V) \cap G(f) \neq \emptyset \). Therefore, \( G(f) \) is not extremely-closed.

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REFERENCES


