FIXED POINT THEOREMS FOR POINT-TO-POINT AND POINT-TO-SET MAPS IN BANACH SPACE

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1. Introduction

Let \( \{x_k\} \) be a sequence in a metric space \((M, d)\). A real nonnegative sequence \( \{t_k\} \) is said to majorize \( \{x_k\} \) if \( d(x_{k+1}, x_k) \leq t_{k+1} - t_k, \) \( k = 0, 1, \ldots \). If \( \lim_{k \to \infty} t_k = t^* < +\infty \) exists, then \( \{x_k\} \) is a Cauchy sequence in \( M \). So if \( M \) is complete, there exists a \( x^* \) in \( M \) such that \( \lim_{k \to \infty} x_k = x^* \) and \( d(x^*, x_k) \leq t^*-t_k, \) \( k = 0, 1, \ldots \); see [10], [12] and [13]. This observation led to a new proof for the Newton-Kantorovich theorem by Ortega [10] and the general convergence theory for Newton related processes by Rheinboldt [13]. Now suppose that we have the problem of solving a nonlinear equation \( F(x) = 0 \) in a Banach space and that an initial point \( x_0 \) for a certain iterative process generates a sequence \( \{x_k\} \). Then the problem of finding roots for \( F(x) = 0 \) is reduced to the construction of a convergence scalar sequence \( \{t_k\} \) which majorizes \( \{x_k\} \). It is our purpose in this paper to formalize fixed point theorems, in the spirit of the above technique, which will have the Banach contraction mapping principle and Browder-Nadler’s fixed point theorem as consequences. Among others, some common fixed point theorems are considered.

2. A majorant from of the Browder-Nadler theorem

Throughout this section \( X \) will be a Banach Space and \( M \) is compact convex subset of Banach space with metric \( \| \cdot \| \).

Let \( CL(M) \) be the family of all closed nonempty subsets of \( M \) endowed with the generalized Hausdorff metric \( D \) induced by \( \| \cdot \| \) [9].

**THEOREM 1.** Let \( G : M \to CL(M) \). Suppose that there exists an isolone function \( \varphi : [0, \infty) \to [0, \infty) \) such that

(i) \( \varphi(t) < t \) for each \( t > 0 \),

(ii) \( D(\|G(x) - G(y)\|) \leq \varphi(\min \{ \|x - y\|, \|x - G(x)\|, \|x - G(y)\|, \|y - G(x)\|, \|y - G(y)\| \} ) \), for each \( x, y \in M \).

Suppose that the sequence \( \{x_k\} \) is defined by iterative process as given \( x_0 \in M \),
\( p \in (0, 1) \) and \( x_k = (1 - p)x_{k-1} + pG(x_{k-1}) \) for all \( n \in \mathbb{N} \), and
\[
\max \{ \| x_{k+1} - x_k \|, \| x_{k+1} - G(x_{k+1}) \|, \| x_{k+1} - G(x_k) \|, \| x_k - G(x_{k+1}) \|, \| x_k - G(x_k) \|, \\
\| G(x_{k+1}) - G(x_k) \| \} \leq \alpha D(\| G(x_k) - G(x_{k-1}) \|), \quad k = 1, 2, \ldots, x_0 \in M.
\]

Furthermore, let the nonnegative real sequence \( \{ t_k \} \) be defined by \( t_{k+1} = t_k + \alpha(\min_{j=1}^{k} \| x_j - x_{k-1} \|) \), \( t_0 = 0 \),
\( t_j \geq \max \{ \| x_1 - x_0 \|, \| x_1 - G(x_1) \|, \| x_1 - G(x_0) \|, \| x_0 - G(x_1) \|, \| x_0 - G(x_0) \|, \\
\| G(x_1) - G(x_0) \| \}, \quad k = 1, 2, \ldots,
\]
converge to \( t^* < +\infty \).

Then \( \{ x_k \} \) converges to a fixed point \( x^* \) of \( G \) with the error estimate
\[
\max \{ \| x^* - x_k \|, \| x^* - G(x^*) \|, \| x_k - G(x^*) \|, \| x_k - G(x_k) \|, \\
\| G(x_k) - G(x^*) \| \} \leq \alpha(1 - t_k), \quad k = 0, 1, \ldots.
\]

**Proof.** We show by induction that:
\[
\max \{ \| x_j - x_{j-1} \|, \| x_j - G(x_j) \|, \| x_j - G(x_{j-1}) \|, \| x_{j-1} - G(x_j) \|, \| x_{j-1} - G(x_{j-1}) \|, \\
\| G(x_j) - G(x_{j-1}) \| \} \leq \alpha(t_j - t_{j-1}), \quad j = 1, 2, \ldots
\]

By assumption for \( j = 1 \) we have:
\[
\max \{ \| x_1 - x_0 \|, \| x_1 - G(x_1) \|, \| x_1 - G(x_0) \|, \| x_0 - G(x_1) \|, \| x_0 - G(x_0) \|, \\
\| G(x_1) - G(x_0) \| \} \leq (t_1 - t_0), \quad j = 1, \ldots, k,
\]
then
\[
\max \{ \| x_{k+1} - x_k \|, \| x_{k+1} - G(x_k) \|, \| x_{k+1} - G(x_{k+1}) \|, \| x_k - G(x_{k+1}) \|, \| x_k - G(x_k) \|, \\
\| G(x_{k+1}) - G(x_k) \| \} \leq \alpha D(\| G(x_k) - G(x_{k-1}) \|).
\]

\( \alpha \Phi(\min \{ \| x_k - x_{k-1} \|, \| x_k - G(x_k) \|, \| x_k - G(x_{k-1}) \|, \| x_{k-1} - G(x_k) \|, \| x_{k-1} - G(x_{k-1}) \|, \\
\| G(x_k) - G(x_{k-1}) \| \} \leq \alpha(1 - t_{k-1}) = \alpha(t_k - t_{k-1}).
\]

Since \( \lim_{k \to \infty} t_k = t^* < \infty \) exists,

the estimate
\[
(1) \quad \max \{ \| x_{k+m} - x_k \|, \| x_{k+m} - G(x_k) \|, \| x_k - G(x_{k+m}) \|, \| x_k - G(x_k) \|, \| G(x_{k+m}) - G(x_k) \| \} \leq \max \sum_{j=k}^{k+m-1} \| x_{j+1} - x_j \|, \sum_{j=k}^{k+m-1} \| x_{j+1} - G(x_j) \|, \sum_{j=k}^{k+m-1} \| x_j - G(x_{j+1}) \|,
\]
\[
\sum_{j=k}^{k+m-1} \| G(x_j) - G(x_{j+1}) \| + \| x_{j+1} - G(x_{j}) \| \leq \sum_{j=k}^{k+m-1} (t_{j+1} - t_j) \leq (t_{k+m} - t_k),
\]
shows that \( \{ x_k \} \) is a Cauchy sequence. By the completeness of \( M \), there exist \( x^* \in M \) such that \( \lim_{k \to \infty} x_k = x^* \).

Suppose that \( \| x^* - G(x^*) \| = \varepsilon > 0 \); and consider
\[
(2) \quad \| x_{k+1} - G(x_k) \| \leq (1 - p)\| x_k - G(x_k) \| + p\| G(x_k) - G(x^*) \|
\leq (1 - p)\| x_k - G(x_k) \| + p \cdot \min \{ \| x_k - x^* \|, \| x_k - G(x_k) \|, \\
\| x_k - G(x^*) \| \},
\]
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We have \( \| x^* - G(x^*) \|, \| x_h - G(x^*) \|, \| x_k - G(x^*) \| \) and \( x_{k+1} - x_k = t(G(x_k) - x_k) \), \( \lim_{k \to \infty} (G(x_k) - x_k) = 0 \). Thus \( \lim_{k \to \infty} G(x_k) = x^* \).

If \( \| x^* - G(x^*) \| > 0 \) then there is \( m \in \mathbb{N} \) such that for all \( n \geq m \), \( n \in \mathbb{N} \).

(3) \( \| x_k - x^* \| < \min \{ \| x^* - G(x^*) \|, \| x^* - G(x_k) \| \} \). It follows from (2) and (3) that for \( n \geq m \),

\[
\| x_{k+1} - G(x_k) \| \leq (1 - \rho) \| x^* - G(x^*) \| + \rho \cdot \min \{ \| x^* - G(x^*) \|, \| x^* - G(x_k) \| \},
\]

\[
\| x^* - G(x^*) \|, \| x^* - G(x_k) \|, \| x^* - G(x^*) \|.
\]

Letting \( k \to \infty \), we have

\[
\varepsilon = \| x^* - G(x^*) \| \leq (1 - \rho) \| x^* - G(x^*) \| + \rho \cdot \min \{ \| x^* - G(x^*) \|, \| x^* - G(x_k) \| \}
\]

\[
< (1 - \rho) \| x^* - G(x^*) \| + \rho \cdot \| x^* - G(x^*) \|
\]

\[
= \| x^* - G(x^*) \| = \varepsilon, \text{ a contradiction.}
\]

Hence \( x^* = G(x^*) \). The estimate follows from (1) as \( m \to \infty \).

COROLLARY 1 (Browder-Nadler [2,8,9]). Let \( G : M \to \mathbb{C}(M) \) and

\[
D(\| G(x) - G(y) \|) \leq q \cdot \min \{ \| x - y \|, \| x - G(x) \|, \| y - G(x) \| \}
\]

for each \( x, y \) in \( M \) and \( q \in (0,1) \) then \( G \) has a fixed point.

PROOF. Since \( 0 < q < 1 \), we have \( \frac{1}{q} > 1 \). Choose \( \alpha > 1 \) in Theorem 1 such that \( qa < 1 \). Let \( x_0 \in M \).

Choose \( x_1 = (1 - \rho)x_0 + \rho G(x_0) \) then there exists \( x_2 \) such that

\[
x_2 = (1 - \rho)x_1 + \rho G(x_1)
\]

such that

\[
\max \{ \| x_2 - x_1 \|, \| x_2 - G(x_2) \|, \| x_2 - G(x_2) \|, \| x_1 - G(x_1) \|, \| x_1 - G(x_2) \|, \| x_1 - G(x_2) \|, \}
\]

\[
\| G(x_2) - G(x_1) \| \leq \alpha D(\| G(x_1) - G(x_2) \|).
\]

Continuing in this way we produce a sequence \( \{ x_k \} \) in \( M \) such that

\[
x_{k+1} = (1 - \rho)x_k + \rho G(x_k)
\]

\[
\max \{ \| x_{k+1} - x_k \|, \| x_{k+1} - G(x_k) \|, \| x_{k+1} - G(x_{k+1}) \|, \| x_{k+1} - G(x_k) \|, \| x_{k+1} - G(x_{k+1}) \|, \| G(x_{k+1}) - G(x_k) \| \} \leq \alpha D(\| G(x_k) - G(x_{k+1}) \|), k = 1, 2, \ldots
\]

Let \( \Phi(t) = q^t \) in Theorem 1, then \( t_{k-1} - t_k = qa(t_{k-1} - t_{k-1}), t_0 = 0 \),

\[
t_1 = \max \{ \| x_1 - x_0 \|, \| x_1 - G(x_1) \|, \| x_1 - G(x_0) \|, \| x_0 - G(x_1) \|, \| x_0 - G(x_0) \|, \| G(x_1) - G(x_0) \| \}
\]

So \( t_k = \sum_{j=0}^{k-1} (qa)^j \cdot \max \{ \| x_1 - x_0 \|, \| x_1 - G(x_1) \|, \| x_1 - G(x_0) \|, \| x_0 - G(x_1) \|, \| x_0 - G(x_0) \|, \| G(x_1) - G(x_0) \| \} \).
and hence $\lim_{k \to \infty} t_k = t^* = \left[ \frac{1}{1-q \epsilon} \right] \cdot \max \{ \| x_1 - x_0 \|, \| x_1 - G(x_1) \|, \| x_1 - G(x_0) \|, \| x_0 - G(x_1) \|, \| x_0 - G(x_0) \|, \| G(x_1) - G(x_0) \| \} < +\infty$

therefore by Theorem 1, $G$ has a fixed point.

If $\Phi$ is continuous, then the solution of $t_{k-1} - t_k = \Phi(\alpha(t_k - t_{k-1}))$, $t_0 = 0$, $k=1, 2, \ldots$, satisfies $\lim_{k \to \infty} t_k = t^* < \infty$, so that $\Phi(0) = 0$.

Making obvious modifications in Theorem 1, we have:

**THEOREM 2.** Let $G : M \to \text{CL}(M)$. Suppose that there exists a continuous and isotone function $\Phi : [0, \infty) \to [0, \infty)$ such that

$$D(\| G(x) - G(y) \|) \leq \Phi(\min \{ \| x - y \|, \| x - G(x) \|, \| y - G(x) \|, \| y - G(y) \| \})$$

for each $x, y$ in $M$.

Let $\alpha < 1$. Suppose that the sequence $\{x_k\}$ is defined by the iterative process as given $x_0 \in M$, $p \in (0, 1)$ and $x_k = (1-p)x_{k-1} + pG(x_{k-1})$ for all $k \in \mathbb{N}$, and

$$\max \{ \| x_{k+1} - x_k \|, \| x_k - G(x_{k+1}) \|, \| x_{k+1} - G(x_k) \|, \| x_k - G(x_{k+1}) \|, \| x_k - G(x_k) \|, \| G(x_{k+1}) - G(x_k) \| \} \leq \alpha D(\| G(x_k) - G(x_{k-1}) \|), \quad k = 1, 2, \ldots, \infty.$$

Furthermore, let the nonnegative real sequence $\{t_k\}$ be defined by

$$t_{k+1} = t_k + \Phi(\alpha(t_k - t_{k-1})), \quad t_0 = 0,$$

$$t_k = \max \{ \| x_1 - x_0 \|, \| x_1 - G(x_1) \|, \| x_1 - G(x_0) \|, \| x_0 - G(x_1) \|, \| x_0 - G(x_0) \|, \| G(x_1) - G(x_0) \| \}$$

for $k = 1, 2, \ldots, \infty$.

Then $\{x_k\}$ converges to a fixed point $x^*$ of $G$ with the error estimate

$$\max \{ \| x^* - x_k \|, \| x^* - G(x^*) \|, \| x - G(x^*) \|, \| x_k - G(x^*) \|, \| x_k - G(x_k) \|, \| G(x^*) - G(x_k) \| \} \leq \alpha(t^* - t_k), \quad k = 0, 1, \ldots$$

**THEOREM 3.** Let $C(M)$ be the family of all nonempty compact subsets of $M$. Let $G : M \to C(M)$. Suppose that there exists an upper right semicontinuous function $\Phi : [0, \infty) \to [0, \infty)$ such that

(i) $\Phi(t) < t$ for each $t > 0$

(ii) $D(\| G(x) - G(y) \|) \leq \Phi(\min \{ \| x - y \|, \| x - G(x) \|, \| y - G(x) \|, \| y - G(y) \| \})$

for each $x, y$ in $M$, then $G$ has a fixed point.

Theorem 3 is a slight generalization of a result of Boyd and Wong [1]. Its proof can be carried over from their proof in [1].
**Remark.** Let $\Phi : [0, \infty) \to [0, \infty)$. Suppose that the nonnegative real sequence $\{t_k\}$ satisfies $t_{k+1} - t_k = \Phi(t_k - t_{k-1})$, $t_0 = 0$, $t_1$ given, $k = 1, 2, \ldots$.

Then the following three conditions, taken together, are not sufficient to imply that $\{t_k\}$ converges.

(a) $\Phi$ is isotone and $\Phi(t) < t$ for each $t > 0$.
(b) $\Phi$ is continuous and isotone.
(c) $\Phi$ is upper semicontinuous from the right and $\Phi(t) < t$ for each $t > 0$.

In fact, let $\Phi(t) = \frac{t}{1+t}$, $t \in [0, \infty)$. Then $\Phi$ satisfies (a), (b) and (c).

Define $\{t_k\}$ by $t_k = \sum_{j=1}^{k} \left( \frac{1}{j} \right)$, $t_0 = 0$, $j = 1, 2, \ldots$. Then $t_{k+1} - t_k = \Phi(t_k - t_{k-1})$, but $\{t_k\}$ is divergent.

**Theorem 4.** Let $\{T_n\}$ be a sequence of (point-to-point) maps from a nonempty Banach space $(M, \| \|)$ into itself. Suppose that for each pair $(T_i, T_j)$ there exists a function $\Phi$ of $[0, \infty) \times [0, \infty) \times [0, \infty) \times [0, \infty]$ into $[0, \infty)$ such that

(i) $\Phi$ is continuous and isotone in each variable, and $\Phi(x, y, x, y) < t$ for all $t > 0$;

(ii) $\|T_i(x) - T_j(y)\| \leq \Phi\left( \min \left\{ \|x - T_i(x)\|, \|y - T_j(y)\| \right\}, \frac{1}{2} \cdot (\|x - T_j(y)\| + \|y - T_i(x)\|, \|x - y\|) \right)$ for all $x, y$ in $M$.

Given $x_0 \in M$, $\rho \in (0, 1)$ and $x_{k+1} = (1 - \rho)x_k + \rho T_{k+1}(x_k)$, $k = 0, 1, \ldots$, $x_0 \in X$.

Assume, further that the sequence $\{t_k\}$ defined $t_{k+1} - t_k = \Phi(t_k - t_{k-1})$, $t_k - t_{k-1}$, $t_{k+1} - t_k$ converges to $t^* < +\infty$, then $\{x_k\}$ converges to the unique common fixed point $x^*$ of $\{T_n\}$ with error estimate

$$\max \left\{ \|x^* - x_k\|, \|x^* - T(x_k)\|, \|T(x_k) - T(x^*)\| \right\} \leq t^* - T_k, \quad k = 0, 1, \ldots$$

**Proof.** We show by induction that $\{t_k\}$ majorizes $\{x_k\}$. By assumption,

$$\max \left\{ \|x_1 - x_0\|, \|x_1 - T(x_0)\|, \|x_0 - T(x_1)\|, \|T(x_1) - T(x_0)\| \right\} \leq t_1 - t_0 \quad \text{and if}$$

$$\max \left\{ \|x_j - x_{j-1}\|, \|x_j - T(x_{j-1})\|, \|x_{j-1} - T(x_j)\|, \|x_{j-1} - T(x_{j-1})\| \right\} \leq t_j - t_{j-1}, \quad j = 1, 2, \ldots, k,$$

then

$$\|x_{k+1} - x_k\| = \|(1 - \rho)x_k + \rho T_{k+1}(x_k) - (1 - \rho)x_k - \rho T_k(x_k)\|$$
Therefore,
\[\|x_{k+1} - x_k\| < \Phi \cdot \min \{\|x_k - x_{k-1}\|, \|x_k - x_{k-1}\|, \|x_k - x_{k-1}\|\} \leq \Phi \cdot \min \{(t_k - t_{k-1}, t_k - t_{k-1}, t_k - t_{k-1})\} = t_{k+1} - t_k.\]

Since \(\lim_{k \to \infty} t_k = t^* < +\infty\), there exists a \(x^*\) in \(M\) such that \(\lim_{k \to \infty} x_k = x^*\) consider.

(2) \[\|x_{k+1} - T_n(x^*)\| \leq (1 - p)\|x_k - T_n(x_k)\| + p\|T_n(x_k) - T(x^*)\| \leq (1 - p)\|x_k - T_n(x_k)\| + p \cdot \min \{\|x_k - x^*\|, \|x_k - T_n(x_k)\|\},\]

and \(x_{k+1} - x_k = (T_n(x_k) - x_k)\), \(\lim_{k \to \infty} (T_n(x_k) - x_k) = 0\). Thus \(\lim_{k \to \infty} T_n(x_k) = x^*\).

If \(\|x^* - T_n(x^*)\| > 0\) then there is \(m \in \mathbb{N}\) such that for all \(n \geq m, n \in \mathbb{N}\).

(3) \[\|x_k - x^*\| < \min \{\|x^* - F_n(x^*)\|, \|x^* - T_n(x_k)\|\} < \|x^* - F_n(x^*)\|,\]

It follows from (2) and (3) that for \(n \geq m\).

\[\|x_{k+1} - T_n(x_k)\| \leq (1 - p)\|x^* - T_n(x^*)\| + p \cdot \min \{\|x_k - x^*\|, \|x_k - T_n(x_k)\|\}, \|x^* - T_n(x^*)\|, \|x^* - T_n(x^*)\|, \|x^* - T_n(x^*)\|\}.\]

Letting \(k \to \infty\), we have
\[\varepsilon = \|x^* - T_n(x^*)\| \leq (1 - p)\|x^* - T_n(x^*)\| + p \cdot \min \{\|x^* - T_n(x^*)\|, \|x^* - T_n(x^*)\|\}, \|x^* - T_n(x^*)\|, \|x^* - T_n(x^*)\|, \|x^* - T_n(x^*)\|\} = \|x^* - T_n(x^*)\| = \varepsilon, \text{ a contradiction.}\]

Hence \(x^* = T_n(x^*)\) for \(n = 1, 2, \ldots\). Suppose that \(x^* \neq \bar{x}\) and \(T_n\bar{x} = \bar{x}\) for each \(n\).

Then
\[0 < \|x^* - \bar{x}\| = \|T_i(x^*) - T_j(x^*)\| \leq \Phi(0, 0, \|x^* - \bar{x}\|, \|x^* - \bar{x}\|) \leq \Phi(\|x^* - \bar{x}\|, \|x^* - \bar{x}\|, \|x^* - \bar{x}\|, \|x^* - \bar{x}\|) \leq \Phi(0, 0, \|x^* - \bar{x}\|, \|x^* - \bar{x}\|) \leq \Phi(0, 0, \|x^* - \bar{x}\|, \|x^* - \bar{x}\|, \|x^* - \bar{x}\|), \text{ a contradiction.}\]

Let \(T_i = T_j = T\) for each \(i, j\) and \(\Phi(a, b, c, d) = q \cdot \min \{a, b, c, d\}, q < 1\).

Then the results of Kannan [5] and Reich [12] can be easily seen to follow.

3. Selfmaps on a compact convex subset of Banach space

**Theorem 5.** Let \((M, \|\|)\) be a nonempty compact convex subset of Banach
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Let $F, G$ be point-to-set maps of $X$ to $\text{CL}(M)$. Suppose that for any distinct $x, y$ in $M$,
\[
D(\|F(x) - G(y)\|) \leq \min \left\{ \frac{1}{2}\|x - F(x)\| + \|y - G(y)\|, \frac{1}{2}(\|x - G(y)\| + \|y - G(x)\|), \|x - y\| \right\},
\]
then $F$ or $G$ has a fixed point.

PROOF. Let $\inf \{\|x - F(x)\| : x \in X\} = r_F$ and $\inf \{\|x - G(x)\| : x \in X\} = r_G$. Then there is a sequence $\{x_n\}$ in $M$ such that $\lim_{n \to \infty} \|x_n - F(x_n)\| = r_F$. As a closed subset of a compact convex subset of Banach space, $F(x_n)$ is compact; so there exists $u_n$ in $F(x_n)$ such that
\[
\|x_n - F(u_n)\| = \|x_n - F(x_n)\|.
\]

By the compactness of $X$, we may, by taking a subsequence, assume that $\{u_n\}$ converges to some $\bar{u}$ in $M$. If there exist some positive integer $p$ such that $n \geq p, x_n = \bar{u}$, then
\[
\|\bar{u} - F(\bar{u})\| \leq \inf_{n \geq k} \|\bar{u} - u_n\|, \quad k = p, p + 1, \ldots.
\]
So
\[
\|\bar{u} - F(\bar{u})\| \leq \sup_{k \geq 1} \inf_{n \geq k} \|\bar{u} - u_n\| = 0.
\]
Thus $\bar{u} = F(\bar{u})$.

Assume then that $x_n = \bar{u}$ for infinitely many $n$'s. By taking a subsequence, we may assume then that $x_n \neq \bar{u}$ for each $n$.

In this case we claim that $\bar{u}$ is a fixed point of $G$.

Suppose not. Since $u_n = F(x_n)$, by the simple fact that $\|x - A\| \leq \|x - y\| + \|y - A\|$ for $x, y \in M$ and $A \subseteq M$, we have
\[
\|\bar{u} - G(\bar{u})\| \leq \|\bar{u} - u_n\| + \|u_n - G(\bar{u})\| \leq \|\bar{u} - u_n\| + \|F(x_n) - G(\bar{u})\|
\]
\[
\leq \inf_{n \geq k} \|\bar{u} - u_n\| + \min \left\{ \frac{1}{2} \|x_n - F(x_n)\| + \|\bar{u} - G(\bar{u})\|, \frac{1}{2}(\|x_n - u_n\| + \|u_n - \bar{u}\|) \right\}
\]
\[
\leq \inf_{n \geq k} \|u_n - \bar{u}\| + \min \left\{ \frac{1}{2} \|x_n - F(x_n)\| + \|\bar{u} - G(\bar{u})\|, \frac{1}{2}(\|x_n - u_n\| + \|u_n - \bar{u}\|) \right\}
\]
\[
= \inf_{n \geq k} \|u_n - \bar{u}\| + \min \left\{ \frac{1}{2} \|u_n - F(x_n)\| + \|\bar{u} - G(\bar{u})\|, \frac{1}{2}(\|x_n - F(x_n)\| + \|u_n - \bar{u}\|) \right\}
\]
\[
= \inf_{n \geq k} \|u_n - \bar{u}\| + \min \left\{ \frac{1}{2} \|u_n - F(x_n)\| + \|\bar{u} - G(\bar{u})\|, \frac{1}{2}(\|x_n - F(x_n)\| + \|u_n - \bar{u}\|) \right\}
\]
Since $\|F(x_n) - \bar{u}\| \leq \|u_n - \bar{u}\|$ for each $n$, $\lim_{n \to \infty} \|F(x_n) - \bar{u}\| = 0$. Therefore, as $n \to \infty$,
we obtain
\[ r_G \leq \| \bar{u} - G(\bar{u}) \| \leq \min \left\{ \frac{1}{2} (r_F + \| \bar{u} - G(\bar{u}) \|), \frac{1}{2} (r_F + \| \bar{u} - G(\bar{u}) \|, r_F) \right\}. \]

Thus
\[ r_G \leq \| \bar{u} - G(\bar{u}) \| \leq r_F. \]

By the same argument we have \( r_F \leq r_G \) and hence \( r_F = r_G = \| \bar{u} - G(\bar{u}) \| \). By the compactness of \( G(\bar{u}) \), there exist \( \bar{u} \neq u^* \). Again, there exists \( v^* = F(u^*) \) such that
\[ \| u^* - v^* \| \leq \| G(\bar{u}) - F(u^*) \| \]
\[ < \min \left\{ \frac{1}{2} (\| \bar{u} - G(\bar{u}) \| + \| u^* - F(u^*) \|), \frac{1}{2} (\| \bar{u} - F(u^*) \| + \| u^* - G(\bar{u}) \|), \bar{u} - u^* \right\} \]
\[ \leq \min \left\{ \frac{1}{2} (\| \bar{u} - G(\bar{u}) \| + \| u^* - F(u^*) \|), \frac{1}{2} (\| \bar{u} - u^* \| + \| u^* - F(u^*) \| + \| u^* - G(\bar{u}) \|), \bar{u} - u^* \right\} \]
\[ \leq \min \left\{ \frac{1}{2} (r_F + \| u^* - v^* \|), \frac{1}{2} (r_F + \| u^* - v^* \|, r_F) \right\}. \]

Since \( \| \bar{u} - G(\bar{u}) \| = r_F \), \( \| \bar{u} - G(\bar{u}) \| = \| u^* - v^* \| \) and \( v^* = F(u^*) \). Thus \( r_F \leq \| u^* - F(v^*) \| \leq \| u^* - v^* \| < r_F \), a contradiction.

So \( \bar{u} = G(\bar{u}) \). Therefore \( F \) and \( G \) has a fixed point. Theorem 5 improves the results in \([3, 4, 6, 14, 15]\) and extends the cases.
\[
\alpha_1(t) + \alpha_2(t) + \alpha_3(t) + \alpha_4(t) + \alpha_5(t) = t
\]
in \([17, \text{Theorem 1 and 2}]\) and
\[
\alpha_1(x, y) + \alpha_2(x, y) + \alpha_3(x, y) + \alpha_4(x, y) + \alpha_5(x, y) = 1
\]
in \([18, \text{Theorem 1 and 2}]\). As an illustration, we give a corollary.

**Corollary 2.** Let \( S, T \) be \((\text{point-to-point})\) self-maps on a nonempty compact convex subset of Banach space \((M, \| \cdot \|)\). Suppose that there exist functions \( \alpha_1 = \alpha_2, \alpha_3 = \alpha_4, \alpha_5 \) from \((0, \infty)\) into \([0, \infty)\) such that
(a) \( \alpha_1(t) + \alpha_2(t) + \alpha_3(t) + \alpha_4(t) + \alpha_5(t) \leq t, t > 0; \)
(b) for any distinct \( x, y \) in \( M \),
\[
\| S(x) - T(y) \| < a_1 \| x - S(x) \| + a_2 \| y - T(y) \| + a_3 \| x - T(y) \| + a_4 \| y - S(x) \| + a_5 \| x - y \|,
\]
where \( a_i = \frac{\alpha_i(\| x - y \|)}{\| x - y \|}, i = 1, 2, \ldots, 5 \), then \( S \) or \( T \) has fixed point. If both \( S \) and \( T \) have fixed points, then each of \( S \) and \( T \) has a unique fixed point and
these fixed points coincide.

PROOF. Let $x, y$ be distinct points in $M$. Since $\alpha_1 = \alpha_2$, $\alpha_3 = \alpha_4$, we have

$$
\|S(x) - T(y)\| \leq \frac{a_1 + a_2}{2} (\|x - S(x)\| + \|y - T(y)\|) + \frac{a_3 + a_4}{2} (\|x - T(y)\| + \|y - S(x)\|) + a_5 \|x - y\|.
$$

$$
\leq \min \left\{ \frac{1}{2} (\|x - S(x)\| + \|y - T(y)\|), \quad \frac{1}{2} (\|x - T(y)\| + \|y - S(x)\|),
\|x - y\| \right\}
$$

By Theorem 1, $S$ or $T$ has a fixed point.

Moreover, suppose that $\bar{x}$ is a fixed point of $S$ and $x^*$ is a fixed point of $T$. Then $x = x^*$, since otherwise

$$
0 < \|\bar{x} - x^*\| = \|S(\bar{x}) - T(x^*)\| < \|\bar{x} - x^*\|, \text{ a contradiction.}
$$

(4) Let $x_{2n+1} = (1 - p)x_{2n} + pS(x_{2n})$, $x_{2n+2} = (1 - p)x_{2n+1} + pT(x_{2n+1})$, $n = 0, 1, \ldots, x_0 \in M$ and $p \in (0, 1)$.

**Theorem 6.** Let $S, T$ be (point-to-point) self-maps on a nonempty compact convex subset of Banach space $(M, \|\|)$. Suppose that for any distinct $x, y$ in $M$,

(5) $\|S(x) - T(y)\| \leq \min \left\{ \frac{1}{2} (\|x - S(x)\| + \|y - T(y)\|), \quad \frac{1}{2} (\|x - T(y)\| + \|y - S(x)\|), \quad \|x - y\| \right\}$. Suppose that $S$ and $T$ are continuous. If $S, T$ have a common fixed point $x^*$, then the iterative procedure (4) converges to $x^*$ for any initial point $x_0$ in $M$.

PROOF. If $x \neq x^*$, then

$$
\|S(x) - x^*\| = \|S(x) - T(x^*)\|
$$

$$
\leq \min \left\{ \frac{1}{2} (\|x - S(x)\|), \quad \frac{1}{2} (\|x - x^*\| + \|x^* - S(x)\|), \quad \|x - x^*\| \right\}
$$

$$
\leq \min \left\{ \frac{1}{2} (\|x - x^*\| + \|S(x) - x^*\|), \quad \|x - x^*\| \right\}.
$$

So $\|S(x) - x^*\| < \|x - x^*\|$ for $x \neq x^*$.

Similarly,

(6) $\|T(x) - x^*\| \leq \|x - x^*\|$ for $x \neq x^*$. By the compactness of $M$, there exists a subsequence $\{x_{j(n)}\}$ converging to $\bar{x}$ in $M$.

If, for some positive integer $k$, $x_k = x^*$, then the result follows.

For example let $k = 2p$; then

$$
x_{2n+1} = (1 - p)x_{2n} + pS(x_{2n})
$$

$$
= (1 - p)x^* + pS(x^*)
$$
and hence \( x_{2n+2} = (1-p)x_{2n+1} + pT(x_{2n+1}) \)
\[ = (1-p)x^* + pT(x^*) \]
\[ = x^* - px^* + px^* \]
\[ = x^*. \]

Assume that for each \( k \), \( x_k \neq x^* \).

Let \( b_n = \|x_n - x^*\| \). Then, by (6), \( \{b_n\} \) is decreasing and therefore converges to some number \( b \) in \([0, \infty)\). Thus every subsequence, say \( \{b_{j(n)}\} \) and \( \{b_{j(n)+1}\} \), converges and has the same limit. If \( b > 0 \), then by the continuity of \( S \) and \( T \).

\[ 0 < \|x-x^*\| = \lim_{n \to \infty} b_n = \lim_{n \to \infty} b_{j(n)} = \lim_{j \to \infty} b_{j(n)+1} = \|S(x) - x^*\| \text{ or } \|T(x) - x^*\|, \]

a contradiction to (6).

**THEOREM 7.** Let \( S, T \) be (point-to-point) self-maps on a nonempty compact convex subset of Banach space \((M, \|\cdot\|)\). Suppose that for each proper closed subset \( K \) of \( M \), \( x, y \in K \), \( x \neq y \), \((S, T)\) satisfies (5). Suppose that \( S \) and \( T \) are continuous. If \( S, T \) have a common fixed point \( x^* \), then the iterative procedure (4) converges to \( x^* \) for any initial point \( x_0 \) in \( M \).

**PROOF.** For any \( x_0 \in M \), define \( A(x_0) = \{x : \|x-x^*\| \leq \|x_0-x^*\|\} \). If \( x \in A(x_0) \) and \( x \neq x^* \), then by (6) \( \|S(x) - x^*\| < \|x - x^*\| \leq \|x_0 - x^*\| \) and \( \|T(x) - x^*\| < \|x - x^*\| \leq \|x_0 - x^*\| \). So \( S \) and \( T \) are self-maps of \( A(x_0) \).

Since \( A(x_0) \) is compact, by Theorem 2, the iterative procedure (4) converges to \( x^* \) for any \( x_0 \) in \( M \).

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Fixed Point Theorems for Point-to-point and Point-to-set Maps in Banach space


