RECURRENTS FOR THE H-FUNCTION OF TWO VARIABLES

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1. Introduction

The $H$-function of two variables occurring in this paper is defined in terms of double Mellin Barnes type contour integral in the following manner [7]

$$H(x, y) = H \left[ \begin{array}{c}
\left\{ \begin{array}{c}
(0, n_1) \\
(p_1, q_1)
\end{array} \right. \\
\left\{ \begin{array}{c}
(m_2, n_2) \\
p_2, q_2
\end{array} \right. \\
\left\{ \begin{array}{c}
(m_3, n_3) \\
p_3, q_3
\end{array} \right. \\
(\alpha_j; \alpha_j', A_j), \beta_j; B_j, \delta_j, E_j, F_j
\end{array} \right]_{x, y}
$$

$$= \left( \frac{1}{2\pi i} \right)^2 \int_{L_1} \int_{L_2} \phi(s, t) \theta_1(s) \theta_2(t) x^s y^t ds dt \quad (1.1)$$

where

$$\phi(s, t) = \prod_{j=1}^{n_1} \Gamma(1 - a_j + \alpha_j s + A_j t) \prod_{j=1}^{q_1} \Gamma(1 - b_j + \beta_j s + B_j t) \prod_{j=1}^{p_1} \Gamma(1 - \alpha_j s - A_j t)$$

$$\theta_1(s) = \prod_{j=1}^{m_2} \Gamma(1 - d_j + \delta_j s) \prod_{j=1}^{n_2} \Gamma(1 - c_j + r_j s)$$

and with $\theta_2(t)$ defined analogously in terms of the parameters $(e_j, E_j), (f_j, F_j)$. An empty product is interpreted as unity, and all the capitalized parameters are positive. The other conditions on the parameters are analogous to those of the $H$-function of one variable, and are given in detail by Gupta and Jain [6].

2. The recurrences

For the sake of simplicity and compactness in the recurrences to follow, we write $H$ for the $H$-function of two variables defined by (1.1) if all its param-
eters are exactly the same as in (1.1). Also, we write $H[d_1+1]$ for the $H$-function of two variables in which $d_1$ is replaced by $d_1+1$ and all other parameters are the same as in (1.1). Again, for that form of the $H$-function of two variables in which $d_1$ is replaced by $d_1+1$, $e_1$ by $e_1-1$ simultaneously, and all other remaining parameters are left unchanged, we use the symbol $H[d_1+1, e_1-1]$, and so on.

**Recurrence Relation 1.**

$$-\partial_{q_k} F_{q_k} H(c_1-1,e_1-1)+r_1 E_1 H[d_1+1,f_1+l]+\partial_{q_k} \left( -E_1 \right) F_{q_k} \times$$

$$H[c_1-1]+F_{q_k} \left( 1-c_1 \right) d_{q_k} \left| H[e_1-1] = \right| -E_1 \left( 1-c_1 \right) F_{q_k} \left| H \right. (2.1)$$

provided that $n_1 \geq 1$, $q_i > m_i$ ($i=2,3$).

**Recurrence Relation 2.**

$$\partial_{q_k} E_1 \left| F_{q_k} \right| H[c_1-1,f_1+1]-F_{q_k} r_1 E_1 H[d_1+1,e_1-1]$$

$$+r_1 E_1 \left| F_{q_k} E_{p_k} \right| H[d_1+1]+f_1 E_{p_k} \partial_{q_k} \left| d_{q_k} \left( c_1-1 \right) \times \left| H \right. (2.1)$$

where

$$m_2 \geq 1, n_3 \geq 1, p_i > n_i, q_i > m_i \left( i=2,3 \right) \quad (2.2)$$

**Proofs.** First of all we note that there are only eleven distinguishable parameters $a_1$, $a_p$, $b_1$, $c_1$, $c_p$, $d_1$, $d_q$, $e_1$, $e_p$, $f_1$ and $f_q$ in the contour integral format for the $H$-function of two variables defined by (1.1). Next, we observe from the definition of the $H$-function of two variables, and the application of the formula $\Gamma(z+1)=z\Gamma(z)$, that the replacement of $a_1$ by $a_1-1$ in (1.1), is equivalent to the introduction of the additional multiplying factor $(1-a_1+c_1=l+A_1t)$ into the contour integral format for $H$. Similarly, the replacement of $a_p$, by $(a_p-1)$ introduces $(a_p-1-c_1=l-A_1t)$, of $b_1$ by $(b_1+l)$ introduces $(-b_1+b_1+l+B_1t)$, of $c_1$ by $(c_1+l)$ introduces $(1-c_1+r_1+l)$, of $c_p$ by $(c_p+l)$ introduces $(c_p-1-r_1+l)$ of $d_1$ by $(d_1+l)$ introduces $(d_1-l+r_1+l)$, of $d_q$ by $(d_q+l)$ introduces $(-d_q+l+r_1+l)$, of $e_1$ by $(e_1+l)$ introduces $(1-e_1+l)$, of $e_p$ by $(e_p+l)$ introdu-
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ces (\( e_\alpha -1-E\beta t \)), of \( f_1 \) by \((f_1+1)\) introduces \((f_1-F_1 t)\) and of \( f_\alpha \) by \((f_\alpha +1)\) introduces an additional multiplying factor \((-f_\alpha +F_\alpha t)\) into the contour integral format for \( H \). The proof of the recurrences 1 and 2 is based upon the observations mentioned above.

To prove the recurrence relation 1 as an illustration, it is clear from what has been said above that the replacement of \( c_1 \) by \((c_1-1)\) and \( e_1 \) by \((e_1-1)\) simultaneously will introduce the factor \((1-c_1+r_1 s) (1-e_1+E_1 t)\): again, the replacement of \( d_\alpha \) by \((d_\alpha +1)\) and \( f_\alpha \) by \((f_\alpha +1)\) simultaneously, will introduce the factor \((-d_\alpha +\delta_\alpha s)(-f_\alpha +F_\alpha t)\) also, the replacement of \( c_1 \) by \((c_1-1)\) alone, will introduce the factor \((1-e_1+E_1 t)\). Consequently, we can form the following five term recurrence relation involving indetermined coefficients \( A, B, C, D \) and \( E \):

\[
AH[c_1-1, e_1-1] + BH[d_\alpha +1, f_\alpha +1] + CH[c_1-1] + DH[e_1-1] = EH \quad (2.3)
\]

and then require that

\[
A(1-c_1+r_1 s)(1-e_1+E_1 t) + B(-d_\alpha +\delta_\alpha s)(-f_\alpha +F_\alpha t) + C(1-c_1+r_1 s)
+ D(1-e_1+E_1 t) = E,
\]

be an identity in \( s, t \) and \( st \). Hence, \( A, B, C, D \) and \( E \) can be evaluated. On evaluating the values of these quantities, we easily arrive at the required recurrence relation 1. Recurrence relation 2 can also be obtained in a similar manner.

3. Special cases

(i) If we specialize the parameters of the various \( H \)-functions of two variables involved in the recurrence relation 1, such that all of them reduce to Kampé de Fériet function [1], we get, by virtue of a known formula [4], after a little simplification, the following recurrence relation involving Kampé de Fériet functions:

\[
-CEF_{0,1}^{0,1}[-: 1+C; 1+E \mid x, y] + (1-D)(1-F)F_{0,1}^{0,1}[-: C; E \mid x, y]
+ (E-F+1) F_{0,1}^{0,1}[-: 1+C; E \mid x, y] + (C-D+1) E F_{0,1}^{0,1}[-: C; 1+E \mid x, y]
\]

\[
= (E-F+1)(C-D+1) F_{0,1}^{0,1}[-: C; E \mid x, y] \quad (3.1)
\]

(ii) Again, if we degenerate all the \( H \)-function of two variables occurring in the recurrence relation 1, to the Fox’s \( H \)-function of one variable by means of a known formula [5], we get, after a little change of notation and simplifica-
tion, the following interesting recurrence relation involving Fox's $H$-function given earlier by Buschman [2, p.41(3)]:

$$\beta_q H[a_1-1]-\alpha_1 H[b_q+1]=-\left|\begin{array}{cc} a_1-1 & b_q \\ \alpha_1 & \beta_q \end{array}\right| H$$

(3.2)

Further, if we degenerate all Fox's $H$-functions involved in (3.2) into Meijer's $G$-functions, we arrive at a known result [3, p.209(11)].

Since, a large number of other special functions of two variables as well as one variable, also follow as special cases of the $H$-function of two variables as mentioned by Mittal and Gupta [7] and Gupta and Jain [6], corresponding recurrences for these functions can also be obtained easily from our main recurrences merely by suitably specializing the parameters in them. We omit detail.

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