A REMARK ON THE SPECIAL CLASSES OF ANALYTIC FUNCTIONS II

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Introduction

There are many classes of analytic functions in the unit disk $U$. In this place, we consider about the special classes $C(k)$, $P^*(\alpha, \beta)$, $D_0(k)$, $G(k)$, $P(k)$, and $R(k)$ of analytic functions in the unit disk $U$. And it is the purpose of this paper to give a relation among these classes.

DEFINITION 1. Let $D(k)$ denote the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

analytic in the unit disk $U$ and satisfying

$$\left| \frac{f'(z)-1}{f'(z)+1} \right| < k \quad (z \in U)$$

for some $k(0 < k \leq 1)$. And let $D_0(k)$ denote the class of analytic and univalent functions $f(z)$ in the class $D(k)$.

For this class $D(k)$, K.S. Padmanabhan showed the following result in [4].

LEMMA 1. Let the function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

belong to the class $D(k)$. Then, we have

$$|f'(z)| \leq \frac{1+k|z|}{1-k|z|}$$

for $z \in U$.

Moreover, S. Owa showed some results for the fractional calculus of functions $f(z)$ in this class $D(k)$ in [2].

DEFINITION 2. Let $R(k)$ denote the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

analytic in the unit disk $U$ and satisfying
for some $k(0 \leq k < 1)$.

For this class $R(k)$, D.B. Shaffer gave some results in [5].

**DEFINITION 3.** Let $P(k)$ denote the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

analytic in the unit disk $U$ and satisfying

$$|f'(z) - \frac{1}{2k}| \leq \frac{1}{2k}$$

for some $k(0 \leq k < 1)$.

For this class $P(k)$, D.B. Shaffer showed the following lemma in [6].

**LEMMA 2.** Let the function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

be in the class $P(k)$. Then, for $z \in U$,

$$\frac{1-|z|}{1+(1-2k)|z|} \leq \Re[f'(z)] \leq \frac{1+|z|}{1-(1-2k)|z|}.$$

**DEFINITION 4.** Let $G(k)$ denote the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

analytic and univalent in the unit disk $U$ and satisfying

$$|f'(z) - 1| < k$$

for some $k(0 \leq k \leq 1)$.

For this class $G(k)$, V. Singh gave some results in [8].

**DEFINITION 5.** Let $C(k)$ denote the class of functions

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0)$$

that are convex of order $k(0 \leq k < 1)$ in the unit disk $U$.

For this class $C(k)$, H. Silverman gave some results in [7].

**DEFINITION 6.** Let $P^*(\alpha, \beta)$ denote the class of functions

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0)$$

analytic and univalent in the unit disk $U$ for which
where $0 \leq \alpha < 1$ and $0 < \beta \leq 1$.

For this class $P^*(\alpha, \beta)$, V.P. Gupta and P.K. Jain showed some results in [1].

Recently, S. Owa showed the following lemma in [3].

**Lemma 3.** Let $0 < k \leq 1$. Then, we have

$$C(1-k) \subset P^*(0, k) \subset D^*(0) \subset R\left(\frac{1-k}{1+k}\right).$$

In particular,

$$C(2-\sqrt{2}) \subset P^*(0, \sqrt{2}-1) \subset D(\sqrt{2}-1) \subset R(\sqrt{2}-1)$$

and

$$C(0) \subset P^*(0, 1) \subset D(1) \subset R(0).$$

**Theorem.** Let $0 < k \leq 1/3$. Then, we have

$$C(1-k) \subset P^*(0, k) \subset D^*_0(k) \subset G\left(\frac{2k}{1-k}\right) \subset P\left(\frac{1}{2}\right) \subset R(0).$$

**Proof.** In the first place, let a function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

be in the class $D^*_0(k)$. Then, we have

$$|f'(z)| \leq \frac{1+k}{1-k}$$

with the aid of Lemma 1. Moreover, from Definition 1,

$$|f'(z) - 1| < k(|f'(z)| + 1).$$

Hence, we have

$$|f'(z) - 1| < k\left(\frac{1+k}{1-k} + 1\right)$$

$$= \frac{2k}{1-k}.$$

Consequently, we have

$$D^*_0(k) \subset G\left(\frac{2k}{1-k}\right)$$

for $0 < k \leq 1/3$.

In the second place, we have

$$G\left(\frac{2k}{1-k}\right) \subset P\left(\frac{1}{2}\right)$$
for $0 < k \leq 1/3$ from Definition 3 and Definition 4. And if the function $f(z)$ belongs to the class $P(1/2)$, by using Lemma 2, we have briefly

$$\text{Re}(f'(z)) > 0,$$

that is,

$$P\left(\frac{1}{2}\right) \subset R(0).$$

Accordingly, we have the relation

$$\mathcal{C}(1-k) \subset P^*(0, k) \subset D_0(k) \subset \mathcal{G}\left(\frac{2k}{1-k}\right) \subset P\left(\frac{1}{2}\right) \subset R(0)$$

for $0 < k \leq 1/3$ with the aid of Lemma 3.

**COROLLARY.** We have the relation

$$\mathcal{C}\left(\frac{2}{3}\right) \subset P^*(0, \frac{1}{3}) \subset D_0\left(\frac{1}{3}\right) \subset \mathcal{G}(1) \subset P\left(\frac{1}{2}\right) \subset R(0).$$

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**REFERENCES**


