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Estimation of Reliability of k-out-of-m Stress-Strength Model in the Independent Exponential Case

*Jae Joo Kim**
*Sung Sup Choi**

ABSTRACT

Suppose a system with m components is subjected to a random stress. We consider the estimation of reliability when data consist of random samples from the stress distribution and the strength distributions. All the distributions are assumed to be independent exponential with unknown scale parameters. An explicit form of system reliability and the minimun variance unbiased estimator are obtained. The asymptotic distribution is also obtained by expanding the minimum variance unbiased estimator about the maximum likelihood estimator and establishing their equivalance. The performance of the two estimators is compared by Monte Carlo Simulation.

1. INTRODUCTION

Suppose that a system consisting of m components, functions if at least $k(1 \leqq k \leqq m)$ components simultaneously operate. The system is subjected to a stress Y which is a random variable with continuous cumulative distribution function (cdf) G. The strengths $X_1, \ldots, X_m$ of the components are independent random variables with continuous cdf's $F_1, \ldots, F_m$, respectively. Then the system reliability, which is the probability that the system does not fail, is given by

$$\begin{aligned} R_{k,m} &= P_r\,(k \text{ or more of } X_1, \cdots, X_m > Y) \\ &= \sum_{j=k}^{m} P_r\,(j \text{ of } X_1, \cdots, X_m > Y) \\ &= \sum_{j=k}^{m} \sum_{\alpha_i} \int_{-\infty}^{\infty} \prod_{i=1}^{j} (1-F_{\alpha_i}(x)) \prod_{i=j+1}^{m} F_{\alpha_i}(x)\, dG(x) \end{aligned} \qquad (1 \cdot 1)$$

Here the sum $\sum_{\alpha_i}$ is taken over all $\binom{m}{j}$ distinct combinations of the integers $\{1,2,\ldots,m\}$ such that exactly j of the $X_i's$ are greater than Y and the remaining $X_i's$ are less than or equal to Y. The particular cases $k=1$ and $k=m$ correspond, respectively, to parallel and series system.

It is assumed that $F_1, \cdots F_m$ and G are exponential distributions with unknown scale parameters and that independent random samples $X_{11}, \ldots, X_{1n_1}, \ldots, X_{m1}, \ldots, X_{mn_m}$ and $Y_1, \ldots, Y_{n_0}$ available from $F_1, \ldots, F_m$ and G, respectively. Strong points in favor of this setting are well explained in (1).

Under these assumptions, we obtain the minimum variance unbiased estimator (MVUE) of the system reliability $R_{k,m}$ given in (1.1). This result can be readily applied to all distributions $(F_1, \ldots, F_m, G)$ having a structural relation of the form

* Department of Computer Science and Statistics, Seoul National University. This work was supported in part by the Research Fund of the Ministry of Education, Korean Government, in 1981.

$$(1-F_1)^{1/\theta_1} = \cdots = (1-F_m)^{1/\theta_m} = (1-G)^{1/\theta_0} = (1-F_0)$$

where F_0 is a known distribution, and $\theta_1, \ldots, \theta_m$ and θ_0 are unknown parameters. With an initial transformation of the data as

$$X'_i = -\log[1-F_0(X_i)], \qquad i=1, \cdots, m$$
$$Y' = -\log[1-F_0(Y)]$$

the problem can then be reduced to one of exponential distributions. For example, Weibull distributions with the same known shape parameter are included in this formulation.

This problem is an extension to nonidentical multicomponent systems of the stress-strength model whose identical component version has been considered by Bhattacharyya and Johnson (1).

2. MVUE of $R_{k,m}$

We first derive an explicit expression for the system reliability $R_{k,m}$ introduced in (1.1) for $(F_1, \cdots, F_m, G)$ belonging to the class of exponential distributions. Suppose that

$$F_i(x) = 1-\exp(-\theta_i x), \qquad i=1, \cdots\cdots, m$$
$$G(x) = 1-\exp(-\theta_0 x), \qquad 0 < x < \infty$$

where all of $\theta_1, \cdots, \theta_m$ and θ_0 are unknown. Then

$$P_r(j \text{ of } X_1, \cdots, X_m > Y)$$
$$= \sum_{\alpha_i} \int_{-\infty}^{\infty} \prod_{i=1}^{j} (1-F_{\alpha_i}(x)) \prod_{i=j+1}^{m} F_{\alpha_i}(x)\, dG(x)$$
$$= \sum_{\alpha_i} \int_0^{\infty} \theta_0 \exp[-\theta_0 x - \sum_{i=1}^{j} \theta_{\alpha_i} x] \prod_{i=j+1}^{m} [1-\exp(-\theta_{\alpha_i} x)]\, dx$$
$$= \sum_{t=j}^{m} \binom{t}{t-j} (-1)^{t-j} \sum_{\alpha_1=1}^{m-t+1} \sum_{\alpha_2=\alpha_1+1}^{m-t+2} \cdots\cdots \sum_{\alpha_t=\alpha_{t-1}+1}^{m} \theta_0 / [\theta_0 + \sum_{i=1}^{t} \theta_{\alpha_i}] \qquad (2\cdot1)$$

From now, the abbreviated notation $\sum_{\alpha_1 \cdots \alpha_t}$ will be used in place of

$$\sum_{\alpha_1=1}^{m-t+1} \sum_{\alpha_2=\alpha_1+1}^{m-t+2} \cdots \sum_{\alpha_t=\alpha_{t-1}+1}^{m}$$

Hence

$$R_{k,m} = \sum_{j=k}^{m} \sum_{t=j}^{m} (-1)^{t-j} \binom{t}{t-j} \sum_{\alpha_1, \cdots, \alpha_t} \frac{\theta_0}{[\theta_0 + \sum_{i=1}^{t} \theta_{\alpha_i}]}$$
$$= \sum_{t=k}^{m} \sum_{j=0}^{k-1} (-1)^{t-j+1} \binom{t}{j} \sum_{\alpha_1, \cdots, \alpha_t} \frac{\theta_0}{[\theta_0 + \sum_{i=1}^{t} \theta_{\alpha_i}]}$$
$$= \sum_{t=k}^{m} C_t \sum_{\alpha_1, \cdots, \alpha_t} \phi_\alpha \qquad (2\cdot2)$$

where $C_t = (-1)^{t+1} \sum_{j=0}^{k-1} \binom{t}{j} (-1)^j$,

$$\phi_\alpha = \frac{\theta_0}{[\theta_0 + \sum_{i=1}^{t} \theta_{\alpha_i}]}$$

Let $T_0 = \sum_{j=1}^{n_0} Y_j$ and $T_i = \sum_{j=1}^{n_i} X_{ij}$, $\quad i=1, \cdots, m$, then the following theorem holds.

Theorem 2.1. The MVUE $\widetilde{R}_{k,m}$ of the system reliability $R_{k,m}$ is given by

$$\widetilde{R}_{k,m} = \sum_{t=k}^{m} C_t \sum_{\alpha_1, \cdots, \alpha_t} (n_0-1) R_0 \int_0^1 \prod_{i=1}^{t} (1-R_{\alpha_i} w)^{n_{\alpha_i}-1} (1-R_0 w)^{n_0-2}\, dw \qquad (2\cdot3)$$

where
$$R_0 = \frac{T^*}{T_0}$$
$$R_{\alpha_i} = \frac{T^*}{T_{\alpha_i}}, \qquad i=1, \cdots, t$$
$$T^* = \min\{T_0, T_{\alpha_1}, \cdots, T_{\alpha_t}\}$$

Proof: We first consider the MVUE of the parametric function

$$\phi_\alpha = \frac{\theta_0}{(\theta_0 + \sum_{i=1}^{t} \theta_{\alpha_i})}$$

for a given choice of $\{\alpha_1, \cdots, \alpha_t\}$. A trivial unbiased estimator of ϕ_α is given by
$g_\alpha(X_{11}, \cdots, X_{m1}, Y_1) = I(X_{\alpha_1 1} > Y_1, \cdots, X_{\alpha_t 1} > Y_1)$

Since $\underline{T}_\alpha=(T_0, T_{\alpha_1}, \cdots, T_{\alpha t})$ is a complete sufficient statistic, the unique MVUE $\widetilde{\phi}_\alpha$ of ϕ_α is the conditional expectation

$$\widetilde{\phi}_\alpha = E[g_\alpha(X_{11}, \cdots, X_{m1}, Y_1) \mid \underline{T}_\alpha]$$
$$= P_r[X_{\alpha_1 1} > Y_1, \cdots, X_{\alpha_t 1} > Y_1 \mid \underline{T}_\alpha]$$

Writing $W_0=\frac{Y_1}{T_0}$, $W_{\alpha_i}=\frac{X_{\alpha_i 1}}{T_{\alpha_i}}$ and

$$V_{\alpha_i}=\frac{T_0}{T_{\alpha_i}}, \quad i=1,\cdots,t,$$ we have

$$\widetilde{\phi}_\alpha=P_r[W_{\alpha_1} > V_{\alpha_1} W_0, \cdots, W_{\alpha_t} > V_{\alpha_t} W_0 \mid \underline{T}_\alpha]$$

We know that the random variables $W_0, W_{\alpha_1}, \cdots, W_{\alpha_t}$ are mutually independent and that the distribution of W_i is beta $(1, n_i-1)$ for $i=0, \alpha_1, \cdots, \alpha_t$.

Furthermore $(W_0, W_{\alpha_1}, \cdots, W_{\alpha_t})$ is independent of $\underline{T}_\alpha$. Thus

$$\widetilde{\phi}_\alpha=\int_0^1 \prod_{i=1}^{t} [1-\min(1, V_{\alpha_i} w)]^{n_{\alpha_i}-1} (n_0-1)(1-w)^{n_0-2}\, dw$$
$$= (n_0-1) R_0 \int_0^1 \prod_{i=1}^{t} (1-R_{\alpha_i} w)^{n_{\alpha_i}-1} (1-R_0 w)^{n_0-2}\, dw. \qquad (2 \cdot 4)$$

Substitution of (2.4) in the linear function (2.2) completes the proof.

For small sample sizes of $n_0, n_1, \ldots, n_m$, the equation (2.3) can be easily reduced to a finite sum. The computation of $\widetilde{R}_{k,m}$ can also be accomplished through the numerical integration for moderate sample sizes. However, for large sample sizes, the computation would be quite labourious and one would look for a reasonable approximation. The asymptotic distribution of $\widetilde{R}_{k,m}$ is investigated in Section 3, where it is shown that, for large sample sizes, the MVUE can be approximated pointwise by the maximum likelihood estimator (MLE), which is easier to compute. We also obtain a first-order correction term for the bias of the MLE.

3. ASYMPTOTIC DISTRIBUTION

Due to the complexity of the expression (2.3), it is very difficult to derive the limiting distribution of $\widetilde{R}_{k,m}$ directly from the expression. To circumvent this difficulty, we consider the asymptotic properties of MLE of $R_{k,m}$ and then establish its asymptotic equivalence with $\widetilde{R}_{k,m}$ In this process, we also obtain a first-order correction term for removal of bias of the MLE.

We denote $\widetilde{R}^{(n)}_{k,m}$ for the MVUE given in (2.3) and $\hat{R}^{(n)}_{k,m}$ for the MLE of $R_{k,m}$ where $n=n_0+\sum_{i=1}^{m} n_i$ is the combined sample size. The MLE of $(\theta_0, \theta_1, \ldots, \theta_m)$ is given by

$$\hat{\theta}_i=\frac{n_i}{T_i}, \qquad i=1,\cdots,m$$

By the invariance property of MLE and the expression (2.1),

$$\hat{R}^{(n)}_{k,m}=\sum_{t=k}^{m} C_t \sum_{\alpha_1 \cdots \alpha_t} \frac{\hat{\theta}_0}{[\hat{\theta}_0+\sum_{i=1}^{t}\hat{\theta}_{\alpha_i}]}$$
$$=\sum_{t=k}^{m} C_t \sum_{\alpha_1 \cdots \alpha_t} \hat{\phi}_\alpha^{(n)} \qquad (3 \cdot 1)$$

The limiting distribution of the MLE is given in the following theorem, where $\underset{\rightarrow}{L}$ denotes the convergence in distribution.

Theorem 3.1. Let $n\to\infty$ such that $\frac{n_i}{n}\to r_i$, $0<r_i<1$, for $i=0,1,\cdots,m$. Then

$$\sqrt{n}\,(\hat{R}^{(n)}_{k,m}-R_{k,m}) \quad \underset{\rightarrow}{L} \quad N(0, \sigma^2_{k,m})$$

where $$\sigma^2_{k,m}=\frac{a_0^2}{r_0\theta_0^2}+\sum_{j=1}^{m}\frac{a_j^2}{r_j\theta_j^2}$$

$$a_0=\sum_{t=k}^{m} C_t \sum_{\alpha_1,\cdots,\alpha_t} \frac{(\sum_{i=1}^{t}\theta_{\alpha_i})}{[\theta_0+\sum_{i=1}^{t}\theta_{\alpha_i}]^2}$$

$$a_j=\sum_{t=k}^{m} C_t \sum^{*}_{\alpha_1,\cdots,\alpha_{t-1}} \frac{\theta_0}{[\theta_0+\theta_j+\sum_{i=1}^{t-1}\theta_{\alpha_i}]^2},$$

$j=1,\cdots,m.$

(Here, $\sum^*_{\alpha_1,\cdots,\alpha_{t-1}}$ is the summation over all possible choice $\alpha_1 < \cdots < \alpha_{t-1}$ from $\{1,\cdots,m\}$ except j.)

Proof. It is well known that

$$\sqrt{n}\left(\frac{1}{\hat{\theta}_i}-\frac{1}{\theta_i}\right) \xrightarrow{L} N(0,\ (r_i\,\theta_i^2)^{-1}), \quad i=0,1,\cdots,m.$$

Since $R_{k,m}$ is a function of $\theta_0, \theta_1, \cdots \theta_m$ with continuous first partial derivatives, the theorem follows from (6a. 2.6), page 387 of Rao (2). To derive an asymptotic distribution of MVUE, we first note that

$$\tilde{R}^{(n)}_{k,m}=\sum_{t=k}^{m} C_t \sum_{\alpha_1,\cdots,\alpha_t} \tilde{\phi}_\alpha,\quad \hat{R}^{(n)}_{k,m}=\sum_{t=k}^{m} C_t \sum_{\alpha_1,\cdots,\alpha_t} \hat{\phi}_\alpha \quad (3\cdot 2)$$

Since the coefficient C_t are fixed constant irrespective of the sample size, it suffices to investigate the relation between $\tilde{\phi}_\alpha$ and $\hat{\phi}_\alpha$ as $n\rightarrow\infty$ and for a given choice of $\{\alpha_1,\cdots,\alpha_t\}$

Theorem 3.2. Let $n\rightarrow\infty$ such that $\frac{n_i}{n}\rightarrow r_i$, $0<r_i<1$ for $i=0,1,\cdots,m$, then we have with probability 1,

$$\tilde{\phi}^{(n)}_\alpha = \hat{\phi}^{(n)}_\alpha - \frac{1}{n}B^{(n)}_\alpha + o\left(\frac{1}{n}\right) \qquad (3\cdot 3)$$

where

$$B^{(n)}_\alpha = \left| \left(\sum_{i=1}^{t}\frac{n_{\alpha_i}}{n}V_{\alpha_i}\right)^2 - \frac{n_0}{n}\left(\sum_{i=1}^{t}V_{\alpha_i}\right)\left(\sum_{i=1}^{t}\frac{n_{\alpha_i}}{n}V_{\alpha_i}\right) - \left(\frac{n_0}{n}\right)^2\sum_{i=1}^{t}V_{\alpha_i} + \frac{n_0}{n}\sum_{i=1}^{t}\frac{n_i}{n}V^2_{\alpha_i}\right| \cdot \left|\frac{n_0}{n}+\sum_{i=1}^{t}\frac{n_{\alpha_i}}{n}V_{\alpha_i}\right|^{-3}$$

Proof : We consider first the two leading terms in an asymptotic expansion of the integral (2.4). Then

$$\tilde{\phi}^{(n)}_\alpha = \frac{n_0}{n}\frac{R_0}{\beta_n} - \frac{R_0}{n}\left|\frac{1}{\beta_n} - \frac{n_0}{n}\frac{\left(\sum_{i=1}^{t}R_{\alpha_i}+2R_0\right)}{\beta_n^2} + \frac{n_0}{n}\frac{\left(\sum_{i=1}^{t}\frac{n_{\alpha_i}}{n}R^2_{\alpha_i}+\frac{n_0}{n}R_0^2\right)}{\beta_n^3}\right| + o\left(\frac{1}{n}\right)$$

where $\beta_n = \frac{1}{n}(n_0 R_0 + \sum_{i=1}^{t} n_{\alpha_i} R_{\alpha_i})$. Recognizing that $\hat{\phi}^{(n)}_\alpha = \frac{n_0}{n}\frac{R_0}{\beta_n}$ and simplifying the expansion, we obtain the result (3. 3).

Letting $B_n = \sum_{t=k}^{m} C_t \sum_{\alpha_1,\cdots,\alpha_t} B^{(n)}_\alpha$,

we have from (3.2) and (3. 3),

$$\tilde{R}^{(n)}_{k,m} = \hat{R}^{(n)}_{k,m} - \frac{1}{n}B_n + o\left(\frac{1}{n}\right) \quad (3.4)$$

By the strong law of large numbers, B_n converges to a finite constant and, hence,

$$\sqrt{n}\ (\tilde{R}^{(n)}_{k,m} - \hat{R}^{(n)}_{k,m}) \rightarrow 0, \qquad \text{almost surely.}$$

Thus the asymptotic distribution stated in Theorem 3. 1 for $\hat{R}^{(n)}_{k,m}$ also holds for $\tilde{R}^{(n)}_{k,m}$. Since $\tilde{R}^{(n)}_{k,m}$ is unbiased for $R_{k,m}$, the term $\frac{1}{n}B_n$ in (3. 4) provides an estimate of the first-order correction term for bias in the MLE.

4. EMPIRICAL COMPARISON WITH MLE

Since the exact distributions of the MVUE and the MLE given in (2.3) and (3.1) are very difficult to obtain analytically, we investigate their relative performance in a moderate sample size $n_0 = n_1 = \cdots\cdots = n_m = 20$ through Monte Carlo simulation. Estimates of the mean squared error (*MSE*) and the bias were obtained from 2,500 trials for one-out-of-three and two-out-of-four systems with various sets of parameters. From these, the value of $\tilde{R}_{k,m}$ was obtained using (2.3) and the value of $\hat{R}_{k,m}$ from (3.1). The true value of $R_{k,m}$ was computed from (2.1).

The results are listed in the table. Although $\tilde{R}_{k,m}$ is known to be unbiased, its estimated bias is recorded as a check on the computation. The estimated bias of MVUE is found to be much smaller in magnitude than that of MLE except for the case $\theta_0 = \theta_1 = \cdots\cdots = \theta_m$. Although the MLE is biased, it is apparent that the magnitude of (biss)^2 is negligible relative the MSE in all cases included in the study. Furthermore, the MSE of both estimators appear to be nearly equal.

Estimates of bias and MSE; $n_0 = n_1 = \cdots\cdots = n_m = 20$

(k, m)	$(\theta_0, \theta_1, \cdots, \theta_m)$	$R_{k,m}$	bias		MSE	
			MVUE	MLE	MVUE	MLE
(1, 3)	(1,1,1,1)	0.75	0.00243	0.00235	0.00487	0.00454
	(3,2,1,1)	0.9286	−0.00608	−0.01008	0.00102	0.00108
	(4,2,2,1)	0.9349	−0.00723	−0.01110	0.00121	0.00128
	(5,3,2,1)	0.9466	−0.00072	−0.00428	0.00062	0.00066
	(8,4,2,1)	0.9748	0.00096	−0.00239	0.00020	0.00022
(2, 4)	(1,1,1,1,1)	0.6	−0.00341	−0.00332	0.00713	0.00677
	(3,2,1,1,1)	0.8536	−0.00099	−0.00666	0.00207	0.00208
	(5,3,2,1,1)	0.8961	−0.00347	−0.00896	0.00163	0.00171
	(8,4,3,2,1)	0.9121	−0.00317	−0.00835	0.00171	0.00179
	(16,8,4,2,1)	0.9599	0.00148	−0.00214	0.00034	0.00038

REFERENCES

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국 문 요 약

m개의 부품으로 이루어진 씨스템이 스트레스를 받고 있다고 가정하자. 본 논문에서는 스트레스 분포와 스트렝스 분포로 부터의 확률표본으로 자료가 구성되어 있을때, 이 씨스템의 신뢰도에 대하여 고찰하였다. 모든 분포는 미지의 모수를 가진 독립적인 지수 분포인 것으로 가정하였다. 씨스템 신뢰도의 형태와 최소분산 불편추정량을 구하였다. 또한, 최소분산 불편추정량을 최우추정량에 대하여 전개하고 두 추정량의 동가성을 보임으로서 근사분포를 구하였다. 몬테카를로 씨뮤레이션으로 두 추정량의 효율을 비교하였다.