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—Research Paper—

Multi-Objective Stochastic Optimization in Water Resources System

Shim. Soon Bo*

Abstract

The purpose of this paper is to present a method of multi-objective, stochastic optimization in water resources system which investigates the development of potential non-normal deterministic equivalents for subsequent use in multiobjective stochastic programming methods. Given probability statement involving a function of several random variables, it is often possible to obtain a deterministic equivalent of it that does not include any orginal random variables.

A Stochastic trade-off technique-MSTOT is suggested to help a decision maker achieve satisfactory levels for several objective functions. This makes use of deterministic equivalents to handle random variables in the objective functions. The emphasis is in the development of non-normal deterministic equivalents for use in multiobjective stochastic techniques.

1. Introduction

This paper presents the subject of continuous random variables in the set of constraints of a stochastic programming problem, to be satisfied under specified probability limits, and presents a large class of deterministic equivalents. These deterministic equivalents no longer contain any of the initial random variables. In the solution Algorithm of the preceding paper, a deterministic equivalent was introduced for the case of a function of normal random variables. In this paper the existence of deterministic equivalents is established for function of continuous random variables with any distribution function.

When the random variables appear in the objective function, the original stochastic problem can be transformed into an equivalent deterministic problem. Given a probability statement involving a function of several random variables, it is often possible to obtain a deterministic equivalent of it that does not include any of the original random variables.

Uncertainty and risks are often formulated as deterministic problems in which the expected values of the random variables of concern are used. The difficulty with this formulation is

* Department of Civil Engineering
Chung Buk National University

that it does not allow the decision-maker to see how the levels achieved for the various objectives vary with the amount of risk he might be willing to take.

These difficulties have provided the motivation to consider the development of a method capable of handling risk in the development of objective trade offs, and yet flexible enough to accomodate the preferences of a decision maker (DM) in a progressive manner in the analysis.

A Stochastic Trade-off Technique-MSTOT is suggested to help a decision maker achieve satisfactory levels for several objective functions. This makes use of deterministic equivalents to handle random variables in the objective functions. The emphasis is in the development of non-normal deterministic equivalents for use in multiobjective stochastic techniques.

2. Statement of the Problem

To state the research problem, we postulate the existence of a decision situation in which there are N resources to be allocated so that P satisfactory goal level may be attained. The main objective is to develop an algorithm which will permit the decision maker(DM) to make a choice among alternative solutions. The following elements are considered;

1. A vector of objective functions

$$\underline{Z}(\underline{x})=(z_1(\underline{x}), z_2(\underline{x}), \cdots, z_p(\underline{x})), \underline{x}\in\underline{X}.$$

Each objective function is defined on a set of resource allocations (the domain) and has values in the set of real numbers (the range). These functions may be either linear or nonlinear, and may contain random parameters. It is required that each objective function be differentiable.

2. A vector of goal functions denoted $\underline{G}$ and defined

$$\underline{G}(\underline{x})=(G_1(\underline{x}), G_2(\underline{x}), \cdots, G_p(\underline{x})) \quad \cdots\cdots (1)$$

Where

$$G_i(\underline{x})=\frac{z_i(\underline{x})-z_i\text{min.}}{z_i(x^*)-z_i\text{min.}}$$

$$z_i(x_i^*)=\text{max. } z_i(x)$$

$$z_i\text{min.}=\underset{\underline{x}\in\underline{X}}{\text{min.}}\ z_i(\underline{x}),\quad i=1, 2, \cdots, P.$$

Each function G is defined on the feasible region $\underline{X}$ and has values in the interval $R(0, 1)$.

3. A set of constraints defining the feasible region $\underline{X}\subset R^m$ and characterized by;

E equality constraints, $\underline{g}(\underline{x})=(g_1(\underline{x}), g_2(\underline{x}), \cdots, g_E(\underline{x}))=\underline{0}$, where $g_i(\underline{x})$ is differentiable, and either linear or nonlinear;

I inequality constraints, $\underline{h}(\underline{x})=(h_1(\underline{x}), h_2(\underline{x}), \cdots, h_I(\underline{x}))\leqslant\underline{0}$, where $h_i(\underline{x})$ is differentiable, and either linear or nonlinear; Q probability constraints of the form

Prob $[r_i(\underline{x}, a_1, \cdots, a_n)\leqslant b_i]\geqslant 1-\alpha_i$, $i=1, 2, \cdots, Q$ where $\alpha_i\in R[0, 1]$, $b_i\in R^+$, $r_i(\underline{x}, a_1, \cdots, a_n)$ is differentiable and either linear or nonlinear. The parameters $a_1, a_2, \cdots, a_n$ are random variables, each with a given probability density function. The functions $g, h,$ and r are defined on the set X with values on an arbitrary set S.

4. A preference function U to articulate the "value structure" of the DM.

This preference function is defined on the set of ranges of the goal functons with values in the interval $R[0, 1]$.

5. An aspiration level for each goal function. This aspiration level is the degree of goal attainment the DM strives to attain. This set of aspiration levels will attempt to identify a subset of the nondominated set.

6. Stochastic parameters in both the objective functions and constraints. The random variables appear in a given objective function, the objective function itself becomes a random variable and the complexity associated with the task of determining the nondominated set is increased. The specification of a value for an objective function is no longer sufficient and, instead one must talk about an achieved value and the probability of achiving that value.

An algorithm is sought, then, to perform a number of sequential tasks and, in the process, take into account the element presented above. This algorithm should identify the nondominated set, order the elements in it according to the DM's preference function, and identify a subset of the nondominated set which satisfies his aspiration levels. Then, as the DM learns, reassesses his preference function, and updates his aspiration levels, the algorithm should be able to redefine the above subset and for each element in it provide probabilities of goal level achivement.

3. The MSTOT Method; The Algorithm Development

In response to that problem statement, this chapter develops a multiobjective algorithm for decision making within the framework of stochastic programming to allow the decision maker to search for alternative solutions.

This multiobjective stochastic trade-off technique, labeled MSTOT, involves the formulation of an initial surrogate objective function (SOF), the estimation of multiattribute utility function reflecting the DM'S preferences, the redefinition of the SOF, and the use of a cutting-plane technique to solve the general nonilnear problem.

In the algorithm itself reference in made to normal random variables.

Using the multiattribute utilty function, weights can be found and used to construct a new surrogate objective function.

The surrogate problem is solved giving a new solution is presented to the DM in the form of a vector of goal values and the probability of attaining those values. If the DM is satisfied the procedure terminates. If not, the least satisfactory objective function and/or probability of attainment is forced to improve. This is done by constructing a deterministic equivalent chance constraint and adding it to the constraint set and removing that objective from the SOF.

The procedure is repeated until a satisfactory solution is found.

In MSTOT, the DM is able to trade the levels of the objective functions and their respective probabilities of achievement against one another.

There are twelve steps in the MSTOT;

1. Problem definition; A vector of objective functions $\underline{Z}(\underline{x})$, and a domain $D_1 \subset R^m$ of admissible solutions is given,

$$\underset{\sim}{Z}(\underset{\sim}{x}) = (z_1(\underset{\sim}{x}), z_2(\underset{\sim}{x}), \cdots\cdots, z_q(\underset{\sim}{x})), \tag{2}$$

$$D_1 = \{\underset{\sim}{x}; \underset{\sim}{x} \in R^m, g_p(\underset{\sim}{x}) \leq 0, \underset{\sim}{x} \geq 0, \ P \in I \ [I, P]\}, \tag{3}$$

$$Z_i'(\underset{\sim}{x}) = \sum_{j=1}^{n} \underset{\sim}{C}_{ij} \ X_j, Z_i(\underset{\sim}{x}) = E[Z_i(\underset{\sim}{x})], \tag{4}$$

$$\underset{\sim}{C}_{ij} \approx N[E(\underset{\sim}{C}_{ij}), \ \mathrm{VAR}(\underset{\sim}{C}_{ij})], \tag{5}$$

and the functions $g_p(\underset{\sim}{x})$ are differentiable and convex.

2. Range of objective functions. Let $\underset{\sim}{X}_i^*$ be such that

$$Z_i(\underset{\sim}{X}_i^*) = \max_{x \in D_1} \cdot Z_i(\underset{\sim}{x}), i \in I[1, q], \tag{6}$$

and define the following,

$$\underset{\sim}{U}_1 = \begin{vmatrix} Z_1(\underset{\sim}{x}_1^*) \\ Z_2(\underset{\sim}{x}_2^*) \\ \vdots \\ Z_q(\underset{\sim}{x}_q^*) \end{vmatrix} \tag{7}$$

$$\underset{\sim}{R} = \{\underset{\sim}{X}_1^*, i \in I[1, q]\} \tag{8}$$

$$Z_{i_{\min}} = \min_{x \in D_1} Z_i(x) \tag{9}$$

3. Initial surrogate objective function. In order that all functions $G_i(x)$ be in [0, 1] let

$$F(\underset{\sim}{x}) = \sum_{i=1}^{q} (G_i(\underset{\sim}{x}), \tag{10}$$

$$\text{where } G_i(\underset{\sim}{x}) = \frac{Z_i(\underset{\sim}{x}) - Z_{i_{\min}}}{Z_i(\underset{\sim}{x}_i^*) - Z_{i_{\min}}}. \tag{11}$$

4. Initial solution; maximize $F(\underset{\sim}{x}), \underset{\sim}{x} \in D_1$. The resulting solution $\underset{\sim}{x}_1$ is then used to generate an initial, nondominated goal vector $\underset{\sim}{G}_1$,

$$\underset{\sim}{G}_1 = \begin{vmatrix} G_1(\underset{\sim}{x}_1) \\ G_2(\underset{\sim}{x}_1) \\ \cdot \\ \cdot \\ \cdot \\ G_q \ (\underset{\sim}{x}_1) \end{vmatrix} \tag{12}$$

5. Utility function choice; A multidimensional utility function $u(\underset{\sim}{G})$ is selected to reflect the DM's goal utility assessment.
The multi-plicative from

$$1 + ku(\underset{\sim}{G}) = \prod_{i=1}^{q} [1 + kk_i u_i(G_i)] \tag{13}$$

is considered for illustrative purposes. The procedure to determine the parameters k, k_i which is presented in those references, will applied here.

6. Redefinition of the surrogate objective function; A new SOF is defined using results of Steps 3 and 5 as follows,

$$S_1(\underset{\sim}{x}) = \sum_{i=1}^{q} W_i G_i(\underset{\sim}{x}) \tag{14}$$

Where $W_i = 1 + \frac{r}{G_i(\underset{\sim}{x}_1)} \left[\frac{\partial U(\underset{\sim}{G})}{\partial G_i} \right]_{\underset{\sim}{G}_1}$ (15)

and r is the step size required to yield a new goal vector in the direction of a desired increment $\varDelta U(\underset{\sim}{G})$. Accordingly,

(a) Computer $U(\underset{\sim}{G}_1)$.

(b) Decide on a value for $0 \leq \varDelta u(\underset{\sim}{G}) \leq 1$.

(c) Solve for the step size r in,

$$\varDelta u(\underset{\sim}{G}) = u(\underset{\sim}{G}_1 + r\nabla u(\underset{\sim}{G}_1) - u(\underset{\sim}{G}_1). \quad (16)$$

7. Generation of alternative solution; Maximize $S_1(\underset{\sim}{x}), \underset{\sim}{x} \in D_1$.

The resulting solution $\underset{\sim}{x}_2$ is then used to generate vectors $\underset{\sim}{G}_2$ and $\underset{\sim}{U}_2$,

$$\underset{\sim}{G}_2 = \begin{vmatrix} G_1(\underset{\sim}{x}_2) \\ G_2(\underset{\sim}{x}_2) \\ \cdot \\ \cdot \\ \cdot \\ G_q(\underset{\sim}{x}_2) \end{vmatrix} \qquad \underset{\sim}{U}_2 = \begin{vmatrix} Z_1\ (\underset{\sim}{x}_2) \\ Z_2\ (\underset{\sim}{x}_2) \\ \cdot \\ \cdot \\ \cdot \\ Z_q(\underset{\sim}{x}_2) \end{vmatrix} \quad (17)$$

8. Generate Vector $\underset{\sim}{V}_1$, which expresses trade-off between goal value and its probability of achievement,

$$\underset{\sim}{V}_1 = \begin{vmatrix} (G_1(\underset{\sim}{x}_2), & 1-\alpha_1) \\ (G_2(\underset{\sim}{x}_2), & 1-\alpha_2) \\ & \cdot \\ & \cdot \\ & \cdot \\ (G_q(\underset{\sim}{x}_2), & 1-\alpha_q) \end{vmatrix} \quad (18)$$

where the element $1-\alpha_i$ is such that,

$$\mathrm{Prob}[Z_i'(\underset{\sim}{x}) \geq Z_i(\underset{\sim}{x}_2)] \geq 1-\alpha_i, \quad (19)$$

or its deterministic equivalent

$$\sum_{j=1}^{n} E(C_{iJ})X_J + K_{\alpha_i}\ [\underset{\sim}{X}^T A\underset{\sim}{x}]^{1/2} \geq Z_i(\underset{\sim}{x}_2) \quad (20)$$

In (30) K_{α_i} is a standard normal value such that $\Phi(K_{\alpha_i}) = \alpha_i$ and Φ represents the cumulative distribution funcion.

The variance-covariance matrtx A is symmetric and positive-definite, and the quadratic form $\underset{\sim}{x}^T\ \underset{\sim}{A}\underset{\sim}{x}$ is then positive-definite. Accordingly $1-\alpha_i \leq 0.5$, so for.

9. The DM now poses the following question: "Are all the $Z_i(\underset{\sim}{x}_2)$ values satisfactory?" In the affirmative case $\underset{\sim}{U}_2$ represents a desired solution. Otherwise, continue.

10. Select the objective function $Z_k(\underset{\sim}{x})$ with the least satisfactory pair $(G_k(\underset{\sim}{x}_2),\ 1-\alpha_k)$ and specify $e_k \in R^+, \alpha_k^0 \in R[0, 1]$, such that

$$\mathrm{Prob}\ [Z'_k(x) \geq e_k] \geq 1-\alpha_k^0. \quad (21)$$

The DM will specify the above if he is not satisfied with either the value achieved for *kth* goal, $G_k(\underset{\sim}{x}_2)$, or the probability of achieving that value, $1-\alpha_k$, or both.

11. Redefine the solution space; Define the new x-space D_2 as follows. $g_p(x) \leq 0$, $p \in I\ [1, P]$,

$$\sum_{j=1}^{n} E(C_{k_j})x_j + K^0_{\alpha_k} \ (\underset{\sim}{x}^T \underset{\sim}{A} \underset{\sim}{x})^{1/2} \geq e_k, \underset{\sim}{x} > 0. \qquad (22)$$

From (22) it is seen that DM is now able to trade directly the value of the *k-th* goal against the probability of achieving such value, as long as the inequality is satisfied.

12. Generate the new surrogate objective function. $S_2(\underset{\sim}{x})$,

$$S_2(\underset{\sim}{x}) = \sum_{i \neq k}^{q} \text{Wi } G_1(\underset{\sim}{x}), \qquad (23)$$

and go back to Step 7 to maximize $S_2(\underset{\sim}{x})$ under D_2.

$S_2(\underset{\sim}{x})$ will contain one term less since the k th objective function now forms part of D_2. Repeat this sequence until a satisfactory vector V_2 is achieved.

$$\underset{\sim}{V}_2 = \left| \begin{matrix} (e_1, & 1-\alpha_1^0) \\ (e_2, & 1-\alpha_2^0) \\ \cdot & \\ \cdot & \\ \cdot & \\ (e_q, & 1-\alpha_q^0) \end{matrix} \right| \qquad (24)$$

By now, the DM has gained considerable information on trade-off between various goal values and the effect of the physical limitations of the problem. The DM is now in a position to reassess his utility function, if he decides to, and go back to step 6 to continue his sequential search for a satisfactum. The random variables presented in the algorithm itself are normal random variables. The algorithm, however, will accomodate any type of random variable, of the continuous or discrete type.

4. Deterministic Equivalents in MSTOT

The general mathematical approach presented here makes use of the change of variable technique (Lindgren, 1968: Hogg and Craig, 1972) to obtain the distribution of a function of several random variables with given distributions. Functions of exponential, uniform, and beta random variables will be considered.

Definition 1. ;

Consider the inequality

$$\sum_{i=1}^{n} \underset{\sim}{C}_i x_i \leq b \qquad (25)$$

where x_i is a mathematical variables, b is a constant, and the $\underset{\sim}{C}_i$ are random variables with known distributions.

Then, the probability statement.

$$\text{Prob.} \left[\sum_{i=1}^{n} \underset{\sim}{C}_i \ x_i \leq b \right] \geq 1 - \alpha \qquad (26)$$

is denoted a chance-constrained inequality and is realized with a minimum probability of $1-\alpha$, where $0 \leq \alpha \leq 1$.

Definition 2. ;

Let a new random variable $\underset{\sim}{y}$ be such that

$$\underset{\sim}{y} = \sum_{i=1}^{n} \underset{\sim}{C}_i x_i \tag{27}$$

with a cumulative distribution function $G(\cdot)$ for y.

Then, the probability statement

$$\text{Prob.}\left[\sum_{i=1}^{n} \underset{\sim}{C}_i\ x_i \leq b\right] > 1 - \alpha \tag{28}$$

is realized if and only if

$$G(b) \geq 1 - \alpha \tag{29}$$

and such inequality is termed a deterministic equivalent of eq(26). Generally, $G(b)$ will be a nonlinear function of the mathematical variables x_i.

When each $\underset{\sim}{C}_i$ is normally distributed with mean $E[\underset{\sim}{C}_i]$ and variance Var $[\underset{\sim}{C}_i]$ and covariance Cov $[\underset{\sim}{C}_i,\ \underset{\sim}{C}_j]$ between $\underset{\sim}{C}_i$ and $\underset{\sim}{C}_j$. The technique suggested by Charnes and Cooper (1963) proceeds as follows:

define $h = \sum_{i=1}^{n} \underset{\sim}{C}_i x_i$,

then h is normally distributed with mean and variance

$E[h] = \sum_{i=1}^{n} E[C_i]\ x_i$,

$\text{Var}\ [h] = X^t \underline{D}_i \underline{X}$

respectively, where

$\underline{X} = (x_1, x_2, \cdots, x_n)^t$

$\underline{D}$ = covariance matrix

$$= \begin{bmatrix} \text{Var}\ (\underset{\sim}{C}_1) & \text{Cov}\ (C_1, C_n) \\ \text{Cov}\ (C_n,\ C_1) & \text{Var}\ (\underset{\sim}{C}_n) \end{bmatrix}$$

Now,

$$P[h \leq b] = P\left\{\frac{h - E[h]}{\sqrt{\text{Var}[h]}} \leq \frac{b - E[h]}{\sqrt{\text{Var}[h]}}\right\}$$

$$= G\left\{\frac{b - E[h]}{\sqrt{\text{Var}[h]}}\right\} \tag{30}$$

where G represents the C.D.F. of a standard normal distribution.

Let K_α be the standard normal value such that $G(K_\alpha) = 1 - \alpha$. Then statement(30) is realized if and only if

$$\frac{b - E[h]}{\sqrt{\text{Var}[h]}} \geq K_\alpha$$

$$\text{or } \sum_{i=1}^{n} E[C_i] X_i + K_\alpha \sqrt{\underline{X}^t\ \underline{D}\underline{X}} \leq b \tag{30}$$

Which is the *deterministic Equivalent of the original stochastic constraint.* The development of non-normal equivalents is now continued with $n = 2$ only to maintain visibility.

Lemma 1. Let $\underset{\sim}{C}_1$ and $\underset{\sim}{C}_2$ be mutually stochastically independent random variables having *uniform distributions* with parameters $(0, b_1)$ and $(0, b_2)$ respectively. Also let x_1, $x_2 \geq 0$, $0 \leq \alpha \leq 1$. Then a deterministic equivalent of the probability statement:

$$\text{Prob}[\underset{\sim}{C}_1 x_1 + \underset{\sim}{C}_2 x_2 \leq d] \geq 1-\alpha \quad (31)$$

where $d \in \text{range}\ (\underset{\sim}{C}_1 x_1 + \underset{\sim}{C}_2 x_2)$, is given by the non-linear inequalities.

$$2(1-\alpha)\ b_1 b_2 x_1 x_2 - d^2 \leq 0 \quad (32)$$

if $0 \leq d \leq b_2 x_2$,

$$2(1-\alpha)\ b_1 x_1 + b_2 x_2 - 2d \leq 0, \quad (33)$$

$$b_2 x_2 < d \leq b_1 x_1.$$

$$2(1-\alpha) b_1 b_2 x_1 x_2 - 2d(b_1 x_2 + b_2 x_2) + d^2 + (b_1 x_1)^2 + (b_2 x_2)^2 \leq 0, \quad (34)$$

$$b_1 x_1 < d \leq b_1 x_1 + b_2 x_2$$

The proof is given in the Appendix A

Lemma 2. Let $\underset{\sim}{C}_1$ and $\underset{\sim}{C}_2$ be mutually stochastically independent random variables having *exponential distributions* with parameters λ_1 and λ_2, respectively. Also, consider the random variable $\underset{\sim}{C}_1 x_1 + \underset{\sim}{C}_2 x_2$, where $x_1, x_2 \in R^+$, and the constant $\alpha \in R[0, 1]$. Then, a deterministic equivalent of the probability statement.

$$\text{Prob}\ [\underset{\sim}{C}_1 x_1 + \underset{\sim}{C}_2 x_2 \leq b] \geq 1-\alpha, \quad (35)$$

where $b \in \text{range}\ (\underset{\sim}{C}_1 x_1 + \underset{\sim}{C}_2 x_2)$, is given by the nonlinear inequality

$$\lambda_1 x_2 \exp\ (-\lambda_2 b / x_2) - \lambda_2 x_1 \exp\ (-\lambda_1 / x_1) - \alpha \lambda_1 x + \alpha \lambda_2 x_1 \leq 0$$

The proof can be obtained in the Appendix A.

The approach is general enough so that developments similar to the ones above can be carried out for functions of any number of independent random variables of the continuous or discrete type, and linear or non-linear in the mathematical variables x_i, but the process may be a lengthy and difficult one.

A weakness of the general approach above is that it is limited to cases where the random variables can be assumed to be stochastically independent with known distribution functions. There are many situations, however, where this assumption may be valid.

In an economic environment, a distributor of, say, energy products, water distribution may realistically consider the contribution margin for each item type to be stochastically independent random variables.

A' Deterministic Transformation

This section addresses the case where some or all of the parameters in the objective function are random variables with known distributions.

Consider the following problem. Let $\underset{\sim}{z}(\underset{\sim}{c}, \underset{\sim}{x})$ be a function of $\underset{\sim}{c} = (\underset{\sim}{c}_1, \underset{\sim}{c}_2, \cdots, \underset{\sim}{c}_n)$, where $\underset{\sim}{c}_i$ is a random variable with distribution $f(c_i)$, and $\underset{\sim}{x} = (x_1, x_2, \cdots, x_n)$, $n \geq 2$, such that $x_i \in R^{++} \cdot \underset{\sim}{z}(\underset{\sim}{c}, \underset{\sim}{x})$ is linear in $\underset{\sim}{c}$.

Also, let $g(\underset{\sim}{x})$ represent an m-column vector of functions of $\underset{\sim}{x}$. Now, suppose the solution to the problem

$$\max\ E[\underset{\sim}{z}(\underset{\sim}{c}, \underset{\sim}{x})] \quad (36)$$

$$\text{s.t.}\ \underset{\sim}{g}(\underset{\sim}{x}) \leq \underset{\sim}{0},$$

yields the solution $\underset{\sim}{x}^*$. Then, the following lemma holds.

Lemma 3.

Given the problem

$$\max\ z(\underset{\sim}{c}, \underset{\sim}{x}) \tag{37}$$
$$\text{s.t. } \underset{\sim}{g}(\underset{\sim}{x}) \leq \underset{\sim}{0},$$
$$F[z(\underset{\sim}{c}, \underset{\sim}{x})] = \alpha,$$

where $z(\underset{\sim}{c}, \underset{\sim}{x}) \in \text{range}\ [\underset{\sim}{z}(\underset{\sim}{c}, \underset{\sim}{x})]$, and $F[\cdot]$ is the cumulative distribution of $z(\underset{\sim}{c}, \underset{\sim}{x})$, there is a unique $\alpha \in R[0, 1]$ for which $(\underset{\sim}{c}^*, \underset{\sim}{x}^*)$ is a solution, and $z(\underset{\sim}{c}^*, \underset{\sim}{x}^*) = E[\underset{\sim}{z}(\underset{\sim}{c}, \underset{\sim}{x}^*)]$. The proof is given in the Appendix A.

An important realization in Lemma 3 is that when random variables are present in the objective function the problem at hand is one in the realm of distribution theory.

The problem, then, becomes that of finding the distribution of the objective function, itself a random variable, in the $(\underset{\sim}{c}, \underset{\sim}{x})$-space. The problem formulation in (37) is termed the *α-model*, for reference convenience, which can now be solved for different values of $\alpha \in R[0, 1]$.

In obtaining the α-model above, we are going from a stochastic, either linear or nonlinear, n-dimensional model to a deterministic, non-linear, m-dimensional model where $m > n$. We note that where as in the stochastic formulation the c's are random variables, in the α-model they become additional mathematical variables constrained by their distribution function.

5. An Illustrative Application

Consider $\underset{\sim}{c}_1$ and $\underset{\sim}{c}_2$ to be exponentially distributed with parameters $\lambda_1 = 1/10$, $\lambda_2 = 1/5$. We would like to know how the new random variable $\underset{\sim}{z} = \underset{\sim}{c}_1 x_1 + \underset{\sim}{c}_2 x_2$ is distributed subject to the constraints

$$x_1{}^2 + x_2{}^2 \leq 25, \tag{28}$$
$$3x_1 + x_2 \leq 12,$$
$$x_1,\ x_2 \geq 0,$$

and for a given value $\alpha \in R[0, 1]$, determine a value $z \in \text{range}\ (\underset{\sim}{z})$ such that

$$\text{Prob.}\ [\underset{\sim}{z} \leq Z] = \alpha.$$

Our α-model can now be formulated with the aid of a deterministic transformation (37) previously stated as follows,

$$\max\ c_1 x_1 + c_2 x_2 \tag{39}$$
$$\text{s.t. } x_1{}^2 + x_2{}^2 \leq 25$$
$$3x_1 + x_2 \leq 12,$$
$$\frac{\lambda_1 \lambda_2}{\lambda_1 \lambda_2 - \lambda_2 x_1} \left[\left(\frac{x_2}{\lambda_2} - \frac{x_1}{\lambda_1} \right) + \frac{x_1}{\lambda_1} e^{-\frac{\lambda_1}{x_1}(c_1 x_1 + c_2 x_2)} - \frac{x^2}{\lambda^2} e^{-\frac{\lambda_2}{x_2}(c_1 x_1 + c_2 x_2)} \right] = \alpha$$
$$c_1,\ c_2 \geq 0,$$
$$x_1,\ x_2 \geq 0.$$

The problem above is *nonlinear and nonconvex*, as can be verified by generating the Hessian matrix of the objective function.

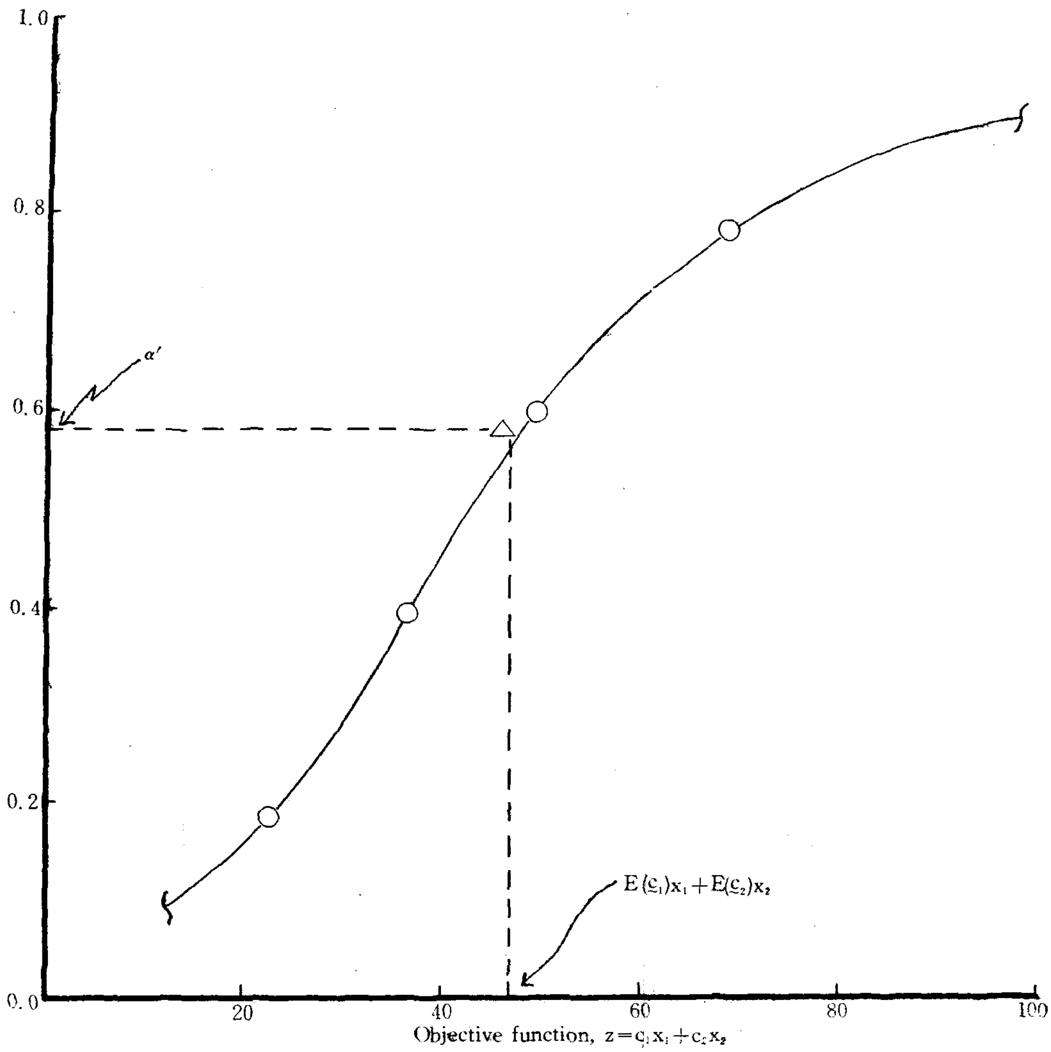
Obtaining a solution to this nonconvex problem is not a trivial matter, as none of the classical techniques can be applied directly here. An approximation technique, however, known as the cutting-plane(Monarchi 1972: Goicoechea, Duckstein and Fogel 1976, a,b) was successfully implemented in computer program SEARCH (Fig. A4 and Appendix B) to solve for the vector (c_1, c_2, x_1x_2) as α was varied parametrically.

The computer results were obtained, and are shown in table 1, and Fig. 1. Now, if the optimization problem is solved using the expected value of the objective function (E-model).

$$\max\ (\underset{\sim}{c}_1)\cdot x_1+E(\underset{\sim}{c}_2)\cdot x_2=10x_1+5x_2 \tag{40}$$
$$\text{s.t. } x_1^2+x_2^2\leq 25,$$
$$3x_1+x_2\leq 12,$$
$$x_1, x_2\geq 0,$$

We obtain the solution $x_1^*=2.570$, $x_2^*=4.289$, with an objective function value $z=47.145$; to obtain the value associated with this solution we evaluate $F(z)$, yielding

Fig. 1. Cumulative distribution of the objective function z.



$$F(z)=F(47.145)=\alpha'=0.586$$

Referring to Fig. 1 we notice that the point (47.145, 0.586) is precisely a single point in the curve obtained solving the α-model problem. The cutting-plane technique used in the solution involves the variable step of size 0.4 which controls the accuracy and rate of convergence. A smaller value for step should produce further accuracy in the results.

Table 1. Computer program results.

α	x_1	x_2	Range of c_1 and c_2 such that $c_1x_1=c_1x_2=z$	Objective function z
0.2	2.234	4.474	$c_1\in[0, 11.126]$	24.857
			$c_2\in[0, 5.555]$	
0.4	2.570	4.289	$c_1\in[0, 14.833]$	38.122
			$c_2\in[0, 8.888]$	
0.6	2.570	4.289	$c_1\in[0, 20.084]$	51.616
			$c_2\in[0, 12.034]$	
0.8	2.570	4.289	$c_1\in[0, 28.478]$	73.190
			$c_2\in[0, 17.064]$	

6. Discussion and Conclusions

It has been shown that it is possible to deal effectively with random variables in the set of constraints and objective function of a stochastic programming problem. When the random variables appear in the set of constraints, deterministic equivalents can now be derived to replace tne original chance-constrained inequality. Thus, the applicability of the multi-objective algorithm MSTOT, is enhanced and extended to a larger, more realistic, variety of problems in water resources management.

When the random variables appear in the objective function the α-model has been structured so that the range of values of the random variable involved is taken into account in the constraint set by means of the cumulative distribution function for those variables, evaluated at a value of $\alpha\in R(0, 1)$. The same concept can be used to deal with probability statements and random variables in the constraint set.

It is important to realize that solution of the stochastic programming problem via an α-model provides all the information that the decision maker would want to extract from such problems, e.g., it completely specifies the magnitude of the objective function as it varies with α, the probability of achievement.

The deterministic equivalents presented in chapter 4 represent exact developments. The change of variable technique used in chapter 4 allows the formulation of exact deterministic equivalents for random variables with any type of probability density function (Goicoechea 1977) thus enhancing the applicability of MSTOT.

It was shown in chapter 4 that it is possible to deal effectively with random variables in the objective function by transforming the original stochastic programming problem into an equivalent deterministic problem.

This new problem formulation, labeled an α-model, allows an analytic, closed-form solution to the stochastic problem whose objective function had been solved previously via simulation. The dimensionality of the problem is increased (generally doubled), and one may go from a linear problem to a nonlinear, nonconvex problem.

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APPENDIX A

Lemma 4. Let $\underset{\sim}{c}_1$ and $\underset{\sim}{c}_2$ be mutually stochastically independent random variables having uniform distributions with parameters $(0, b_1)$ and $(0, b_2)$, respectively. Also let $x_1, x_2 > 0$. Then the new random variable $\underset{\sim}{y} = \underset{\sim}{c}_1 x_1 + \underset{\sim}{c}_2 x_2$ is distributed as follows:

$$
\begin{aligned}
g(y) &= y/b_1 b_2 x_1 x_2, \quad 0 \leq y \leq b_2 x_2 \qquad \text{(A1)}\\
&= 1/b_1 x_1, \quad b_2 x_2 < y \leq b_1 x_1\\
&= \frac{b_1 x_1 + b_2 x_2 - y}{b_1 b_2 \; x_1 x_2}\\
&\quad b_1 x_1 < y \leq b_1 x_1 + b_2 x_2\\
&= 0, \quad \text{otherwise.}
\end{aligned}
$$

Proof: The joint distribution of $\underset{\sim}{c}_1$ and $\underset{\sim}{c}_2$ is given by

$$
\begin{aligned}
f\;(c_1, c_2) &= \left(\frac{1}{b_2}\right)\left(\frac{1}{b_2}\right) \text{for } (c_1, c_2) \in\\
A &= \{(c_1, c_2):\; 0 < c_i \leq b_i,\; i = 1, 2\} \qquad \text{(A2)}\\
&= 0, \quad \text{otherwise;}
\end{aligned}
$$

now, the transformation $y_1 = c_1 x_1 + c_2 x_2$, $y_2 = c_2$ maps the set A into the set B defined as follows

$$
\mathrm{B} = \{(y_1, y_2): 0 \leq y_2 \leq b_2,\; y_2 \leq y_1/x_2,\; 0 \leq y_1 \leq x_1 b_1 + x_2 y_2\}, \qquad \text{(A3)}
$$

as shown in Figures 1 and 2 below,

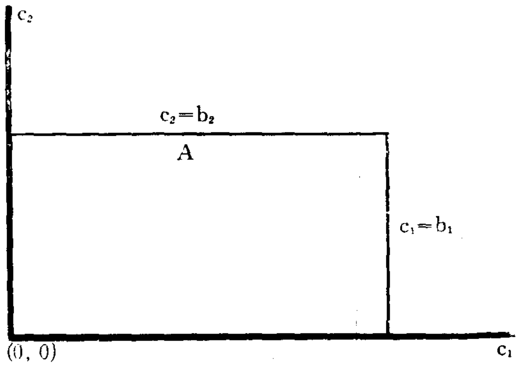

Fig. A1. The set A in the $C-C$ plane.

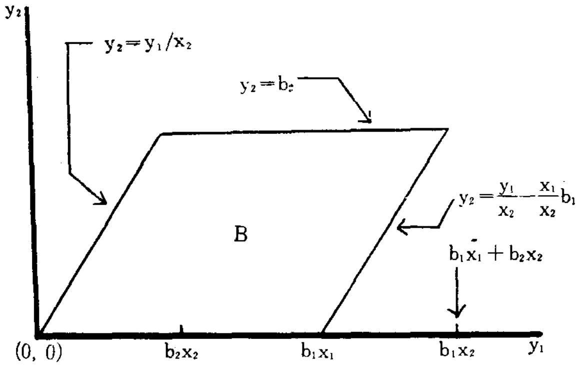

Fig. A2. The set B in the $y_1 - y_2$ plane.

The jont distribution of y_1 and y_2 is then, given by

$$g(y_1, y_2) = \frac{1}{b_1 b_2 x_1}, \text{ for } (y_1, y_2) \in B \tag{A4}$$

$$= 0, \text{ otherwise,}$$

and finally, the pdf $g(y_1)$,

$$g(y_1) = \int_0^{y_1/x_2} g(y_1, y_2) \; dy_2,$$

$$0 \leq y_1 \leq b_2 x_2$$

$$g(y_1) = y_1 / b_1 b_2 x_1 x_2, \tag{A5}$$

$$= \int_0^{b_2} \frac{1}{b_1 b_2 x_1} dy_2,$$

$$= 1/b_1 x_1, \; b_2 x_2 < y_1 \leq b_1 x_1 = \int_{\frac{y_1}{x_2} - \frac{x_1}{x_2} b_1}^{b_2} \frac{1}{b_1 b_2 x_1} dy_2,$$

$$= \frac{b_1 x_1 + b_2 x_2 - y_1}{b_1 b_2 x_1 x_2}$$

$$b_1 x_1 < y_1 \leq b_1 x_1 + b_2 x_2$$

$$= 0, \text{ otherwise;}$$

whenever $b_2 x_2 \geq b_1 x_1$, the definition of $g(y_1)$ above can be modified accordingly.

Lemma 5. Let c_1 and c_2 be mutually stochastically independent random variables having uniform distribution with parameters $(0, b_1)$ and $(0, b_2)$, respectively. Also let $x_1, x_2 \geq 0$, $0 \leq \alpha \leq 1$. Then, a deterministic equivalent of the probability statement

$$\text{Prob } \{c_1 x_1 + c_2 x_2 \leq d\} \geq 1 - \alpha \tag{A6}$$

where $d \in$ range $(c_1 x_1 + c_2 x_2)$, is given by the non-linear inequalities

$$2(1-\alpha) \; b_1 b_2 x_1 x_2 - d^2 \leq 0,$$

$$\text{if } 0 \leq d \leq b_2 x_2,$$

$$2(1-\alpha) \; b_1 x_1 + b_2 x_2 - 2d \leq 0,$$

$$b_2 x_2 < d \leq b_1 x_1,$$

$$2(1-\alpha) \; b_1 b_2 x_1 x_2 - 2d(b_1 x_1 + b_2 x_2) + d^2 + (b_1 x_1)^2 + (b_2 x_2)^2 \leq 0,$$

$$b_1 x_1 < d \leq b_1 x_1 + b_2 x_2$$

Proof: From Lemma 1 the distribution $g(y)$ of the random variable $y = c_1 x_1 + c_2 x_2$ is given by

(31), (32) and (34). The cumulative distribution $G(y)$ is then given as follows,

$$G(y)=\int_0^y \frac{y}{b_1b_2x_1x_2}dy, \quad \text{(A7)}$$

$$=\frac{y^2}{2b_1b_2x_1x_2}, \quad 0\leq y\leq b_2x_2$$

$$=\int_{b_2x_2}^y \frac{1}{b_1x_1}dy+\frac{(b_2x_2)^2}{2b_1b_2x_1x_2},$$

$$=\frac{2y-b_2x_2}{2b_1x_1}, \quad b_2x_2<y\leq b_1x_1$$

$$=\int_{b_1x_1}^y \frac{(b_1x_1+b_2x_2)-y}{b_1b_2x_1x_2}dy+\frac{2b_1x_1-b_2x_2}{2b_1x_1}$$

$$=\frac{1}{2b_1b_2x_1x_2}[2(b_1x_1+b_2x_2)y-y^2-(b_1x_1)^2-(b_2x_2)^2]$$

for $b_1x_1<y\leq b_1x_1+b_2x_2$,

then, the probability statement

$$\text{Prob }[c_1x_1+c_2x_2\leq d]\geq 1-\alpha \quad \text{(A8)}$$

is realized if and only if $G(d)\geq 1-\alpha$, or

$2(1-\alpha)\ b_1b_2x_1x_2-d^2\leq 0,\ \ 0\leq d\leq b_2x_2$

$2(1-\alpha)\ b_1x_1+b_2x_2-2d\leq 0,\ \ b_2x_2<d\leq b_1x_1$

$2(1-\alpha)\ b_1b_2x_1x_2-2d(b_1x_1+b_2x_2)+d^2+(b_1x_1)^2+(b_2x_2)^2\leq 0,$

$b_1x_1<d\leq b_1x_1+b_2x_2.$

APPENDIX B

A CUTTING-PLANE TECHNIQUE

As shown in Chapters 5 and 6 the reduction of a stochastic problem to a deterministic equivalent results in added complexity and, generally, the problem cannot be solved by applying the simplex or dual simplex methods (Dantzig 1963; Luenberger 1973) alone. A nonlinear technique due to Kelly (1960) known as a cutting-plane method was used to effectively solve the nonlinear problem.

Two computer routines have been developed at least (Griffith and Stewart 1960; Monarchi et al. 1973) with some experience detailed. More recent activity (Goicoechea, Duckstein and Fogel 1976a,b) presents a modified version of the method to avoid cycling and speed-up convergence to a solution. Essentially, this method replaces the original constraint set by a set of half spaces. These half spaces are updated progressively to "cut away" portions of the new constraint set. In the process, linear programming is applied to iteratively arrive at a solution in the original feasible region.

Mathematically the nonlinear problem may be stated as follows:

$$\max G=\sum_{j=1}^{n} c_jx_j+h\ (y_1, y_2, \cdots, y_k)$$

subject to

$$\sum_{j=1}^{n} a_{ij}+g_i(y_1, y_2, \cdots, y_k)\leq b_i, \quad i=1, 2, \cdots, m$$

$$L_p \leq y_p \leq U_p, \qquad p=1, 2, \cdots, k$$

$$x_j \geq 0 \qquad j=1, 2, \cdots, n.$$

The problem may now be linearized in the region about the point $x_j=0$, $y_i=y_i^0$ by expansion as a Taylor's series and ignoring terms or order higher than one. The linearized form is,

Maximize (or minimize):

$$G'=\sum_{j=1}^{n} c_j x_j + g_1(y_1^0, y_2^0, \cdots, y_k^0) + \sum_{r=1}^{k} \frac{\partial g_1(y_1^0, y_2^0, \cdots, y_k^0)}{\partial y_r}(y_r - y_r^0)$$

subject to:

$$\sum_{j=1}^{n} a_{ij} x_j + g_i(y_1^0, y_2^0, \cdots, y_k^0) + \sum_{r=1}^{k} \frac{\partial g_i(y_1^0, y_2^0, \cdots, y_k^0)}{\partial y_r}(y_r - y_r^0) = b_i$$

$$L_p - y_p^0 \leq y_r - y_p^0 \leq U_p - y_p^0, \qquad p=1, 2, \cdots, k$$

and $\quad x_j \geq 0, \qquad j=1, 2, \cdots, n$

also, the restriction

$$\Delta^+ y_i + \Delta^- y_i \leq m_i$$

is added to control the convergence rate with step size m_i.

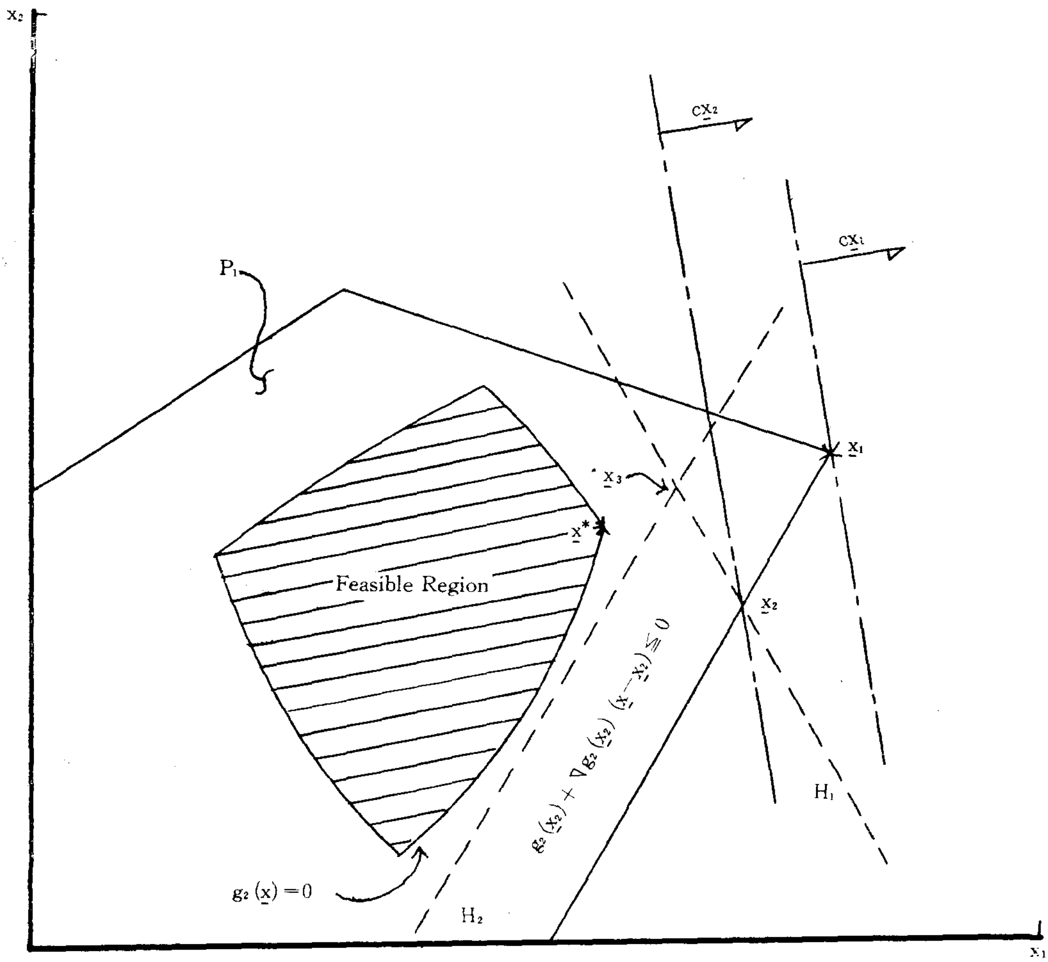

Fig. A3. Cutting plane method.

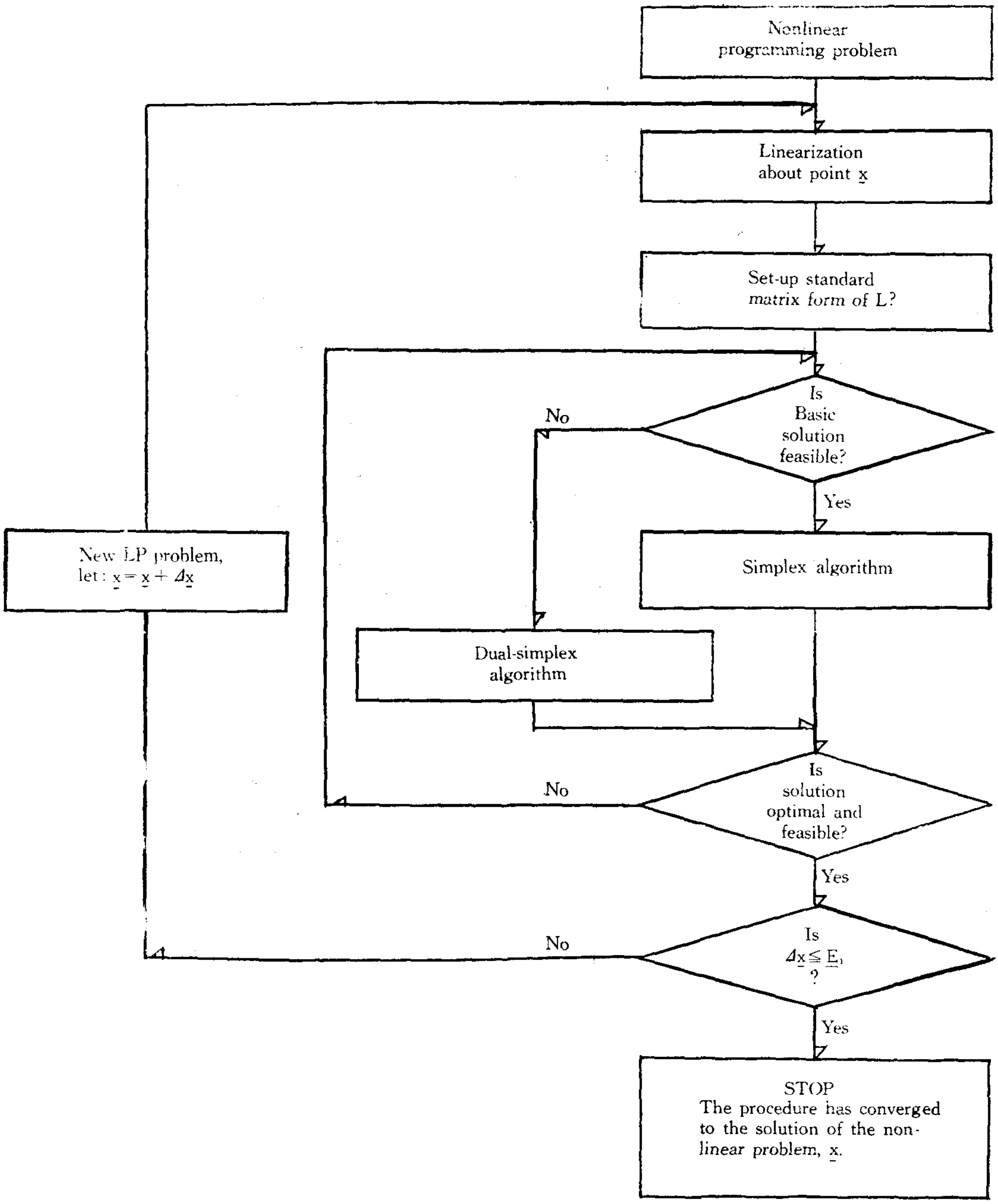

Fig. A4. Program SEARCH, the optimization algorithm.

If $\Delta y_i = y_i - y_i^0$, then $\Delta^+ y_i = \Delta y_i$ when $\Delta_i \geq 0$ and

$\Delta^- y_i = -\Delta y_i$ when $\Delta y_i = 0$.

The linearization procedure and optimization algorithm are illustrated in Figures $A-1$ and $A-2$, respectively. A numercial example is also presented for illustrative purposes.

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