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A Note on the Continuity of the Nearest Point Map

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It was shown by Halpern [2] that if V is a semi-strictly convex, compact convex subset of a normed linear space X then the nearest point map $N: X\to V$ given by $N(x)$ =the unique $y\in V$ such that $\|x-y\| = \inf_{z\in V}\|x-z\|$ is a well defined continuous map such that $N(x)=x$ for all $x\in V$. In this note, we shall show that the compactness condition on V can be weakened to that of approximative compactness, a notion introduced by Efimov and Steckin [1].

To start with, we recall a few definitions.

Definition 1 [4]. Let X be a linear space. If $x\in X$, $y\in X$, $x\neq y$ then *intv* xy denotes the set of all points of the form $\alpha x+\beta y$, $\alpha>0$, $\beta>0$, $\alpha+\beta=1$.

Definition 2 [4]. Let V be a subset of a linear space X. A point $x\in V$ is said to be a *core point* of V if for each point $y\in X$, with $y\neq x$, there exists a point $z\in$ intv xy such that $xz\subset V$. The set of all core points of V is called the *core* V.

Definition 3 [2]. A subset V of a linear space X is said to be *semi-strictly convex* if $x, y\in V$ and $0<t<1$ imply $(1-t)x+ty\in$ core V.

Definition 4 [1]. A subset V of a metric space (X, d) is said to be *approximatively compact* if for every $x\in X$ and every sequence $\langle g_n\rangle$ in V with $\lim_{n\to\infty} d(x, g_n)=d(x, V)$ there exists a subsequence $\langle g_{n_k}\rangle$ converging to an element of V.

The following theorem which gives a set of conditions under which the nearest point map is continuous generalizes a result of Halpern [2].

Theorem. *Let V be a semi-strictly convex approximatively, compact convex subset of a normed linear space X. Then the nearest point map $N: X\to V$ given by $N(x)$ =the unique $y\in V$ such that $\|x-y\| = \inf_{z\in V}\|x-z\|$ is a well defined continuous map such that $N(x)=x$ for all $x\in V$.*

Proof. Let $x\in X$. Since V is approximatively compact, it is proximinal *i.e.* there exists at least one point $y\in V$ such that $\|x-y\|=\inf_{z\in V}\|x-z\|$ (Theorem 2.1, p. 382 [3]). If $x\in V$, then $y=x$ is unique. Suppose $y'\neq y$ were another such point. Consider $z=\frac{1}{2}y+\frac{1}{2}y'\in V$. By the semi-strict convexity, $z\in$ core V. So there exists $z'\in$ intv zx such that $zz'\subset V$, where

$$zz'=\{\alpha z+\beta z' : \alpha\geqslant 0,\ \beta\geqslant 0,\ \alpha+\beta=1\}.$$

So $z'\in V\cap\{tz+(1-t)x : 0<t<1\}$ and thus

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$$\|x - z'\| < \|x - z\| = \left\| \frac{1}{2}(x - y) + \frac{1}{2}(x - y') \right\|$$
$$\leqslant \frac{1}{2}\|x - y\| + \frac{1}{2}\|x - y'\| = \inf_{q \in V}\|x - q\|$$

which is not true. Thus y is unique and so $N: X \to V$ is well defined and also V is Chebyshev.

Since V is an approximatively compact, Chebyshev set, N is continuous (Theorem 4.1, p. 390 [3]). Obviously, $N(x) = x$ for all $x \in V$ and thus the proof is complete.

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