CONFORMAL CHANGE IN 2-DIMENSIONAL UNIFIED FIELD THEORY

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I. Introduction

IA. Two dimensional unified field theory (2-g-UFT): In the usual Einstein's 2-g-UFT the generalized 2-dimensional Riemannian space $X_2$, referred to a real coordinate system $x^i$, is endowed with a real nonsymmetric tensor $g_{ij}$ which may be split into its symmetric part $h_{ij}$ and skew-symmetric part $k_{ij}$ (*):

\begin{equation}
\begin{aligned}
g_{ij} &= h_{ij} + k_{ij}, \\
\text{where}
\end{aligned}
\end{equation}

where

\begin{equation}
\begin{aligned}
(1.1a) & g = \text{Det}(g_{ij}) \neq 0, \quad h = \text{Det}(h_{ij}) \neq 0, \\
(1.1b) & f = \text{Det}(f_{ij}) = (k)_{ij}^2 \neq 0.
\end{aligned}
\end{equation}

We may define a unique tensor $h^{ij}$ by

\begin{equation}
\begin{aligned}
(1.2) & h_{ij}h^{ij} = \delta^i_j.
\end{aligned}
\end{equation}

In 2-g-UFT we use both $h_{ij}$ and $h^{ij}$ as tensors for raising and/or lowering indices of all tensors defined in $X_2$ in the usual manner.

The densities defined in (1.1)b are related by

\begin{equation}
\begin{aligned}
(1.3a) & g = b + f, \\
(1.3b) & g = 1 + k,
\end{aligned}
\end{equation}

so that

\begin{equation}
\begin{aligned}
(1.3c) & g = b/k, \quad k = f/h.
\end{aligned}
\end{equation}

The following tensors will be used in our further considerations:

\begin{equation}
\begin{aligned}
(1.3d) & (\omega)k_i^\nu = \delta_i^\nu, \quad (p)k_i^\nu = (p-1)k_i^\alpha k^\nu_{\alpha}, \quad (p = 1, 2, \ldots).
\end{aligned}
\end{equation}

The differential geometric structure is imposed on $X_2$ by the tensor $g_{ij}$ by means of a connection $\Gamma_{ij}^\nu$ given by the following system of Einstein's equations

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(*) Throughout the present paper, all indices take values 1, 2 and follow the summation convention with the exception of indices $x, y, z$. Greek indices are used for the holonomic components of a tensor and Roman indices for the nonholonomic components.
(1.4) \[ D_\alpha g_{\beta \mu} = 2 S_{\mu \rho}^\alpha g_{\beta \rho} \]
where \( D_\alpha \) is the symbolic vector of the covariant derivative with respect to \( \Gamma_{\lambda \mu}^\nu \) and

(1.5) \[ S_{\lambda \rho}^\nu = \Gamma_{\lambda \rho}^\nu. \]

We note from the last condition of (1.1)b that there exists only the first class of \( k_{\lambda \rho} \) in 2-\( g \)-UFT.

II. Purpose. The conformal change of \( g_{\lambda \mu} \) was primarily studied in \( X_4 \) by Hlavaty ([1], p.151). The purpose of the present paper is to investigate how the conformal change enforces the connection in 2-\( g \)-UFT and to give the complete relations between connections in terms of \( g_{\lambda \mu} \).

II. Recurrence relations in \( X_2 \)

In this section, the recurrence relations in \( X_2 \), obtained by Chung ([2]), will be briefly introduced without proof. These relations will be needed in our further considerations.

Throughout the present paper we use the following Mishra's abbreviations, denoting an arbitrary tensor \( T_{\alpha \mu \nu} \) by \( T([3]) \):

(2.1)a \[ \tilde{A}_{\alpha \beta \rho}^\mu \equiv (x)^\nu K^\alpha_\alpha (x)^\beta K^\rho_\beta (x)^y K^\gamma_\gamma, \]
(2.1)b \[ T = T_{\alpha \mu \nu} = \tilde{A}_{\alpha \beta \rho}^\mu T_{\beta \rho \nu}, \quad T = T_{\alpha \mu \nu} = T. \]

If the tensor \( T_{\alpha \mu \nu} \) is skew-symmetric in the first two indices, then we have

(2.2) \[ T_{\alpha \rho \nu} = -q_{\alpha \rho \nu}. \]

IIA. The first recurrence relations. We have

(2.3) \[ (\rho + 2)k^\alpha_{\lambda \mu} + k^\alpha_{\lambda \mu} = 0, \quad (\rho = 0, 1, 2, \ldots). \]

IIB. The second recurrence relations. If \( T_{\alpha \mu \nu} \) is a tensor skew-symmetric in the first two indices, we have

(2.4)a \[ (10)_{r} T^{(0)}_{\alpha \beta \gamma} = 0, \quad (r = 0, 1, 2, \ldots). \]
(2.4)b \[ T = k T. \]

IIC. The third recurrence relations. If \( T_{\alpha \mu \nu} \) is a tensor skew-symmetric in the first two indices, then we have

(2.5)a \[ (10)^{(0)}_{r} T_{\alpha \beta \gamma}^{(0)} = 0, \quad (r = 0, 1, 2, \ldots) \]
(2.5)b \[ T_{\alpha \beta \gamma}^{(0)} T_{\alpha \beta \gamma}^{(0)} = k T_{\alpha \beta \gamma}^{(0)}. \]
IIID. If $T_{\mu \nu}$ is a tensor skew-symmetric in the first two indices, then we have

$$T_{(\mu \nu)} = 0.$$  

III. Conformal Change in 2-g-UFT

Consider two spaces $X_2$ ($\bar{X}_2$), on which the differential geometric structure is imposed by a general real tensor $g_{\lambda \mu}$ ($\bar{g}_{\lambda \mu}$) through the connection $\Gamma^\nu_{\lambda \mu}$ ($\bar{\Gamma}^\nu_{\lambda \mu}$) defined respectively (1.4) and

$$D_\omega \bar{g}_{\lambda \mu} = 2 \bar{S}_{\omega \mu}^\alpha \bar{g}_{\lambda \alpha}.$$  

REMARK. In our subsequent considerations, we agree that, if $T$ is a function of $g_{\lambda \mu}$, then we denote by $\bar{T}$ the same function of $\bar{g}_{\lambda \mu}$. In particular, if $T$ is a tensor, so is $\bar{T}$. Furthermore, the indices of $T$ (of $\bar{T}$) will be raised and/or lowered by means of $h_{\lambda \mu}$ and/or $\bar{h}^{\lambda \mu}$ (by means of $\bar{h}_{\lambda \mu}$ and/or $\bar{h}^{\lambda \mu}$).

We say that $X_2$ and $\bar{X}_2$ are conformal if and only if

$$\bar{g}_{\lambda \mu}(x) = \exp[\Omega(x)] g_{\lambda \mu}(x),$$  

where $\Omega = \Omega(x)$ is an arbitrary function of position with at least two derivatives. This conformal change enforces a change of the connection, and it can always be expressed as follows ([1], p. 151):

$$\bar{\Gamma}^\nu_{\lambda \mu} = \Gamma^\nu_{\lambda \mu} + Q^\nu_{\lambda \mu},$$

or equivalently

$$\bar{\Gamma}^\nu_{\lambda \mu} = \Gamma^\nu_{\lambda \mu} + M^\nu_{\lambda \mu} + N^\nu_{\lambda \mu},$$

where

$$M^\nu_{\lambda \mu} = Q(\lambda \mu)^\nu, \quad N^\nu_{\lambda \mu} = Q(\lambda \mu)^\nu.$$  

The main purpose of the present paper is to express the tensors $M^\nu_{\lambda \mu}$ and $N^\nu_{\lambda \mu}$ in terms of $g_{\lambda \mu}$.

THEOREM 3.1. We have

$$\Omega_\omega h_{\lambda \mu} = 2 N_{(\lambda \mu, \omega)} + 2 M_{\omega [\lambda \mu]},$$  

$$\Omega_\omega k_{\lambda \mu} = 2 M_{\omega [\lambda \mu]} + 2 N_{(\lambda \mu, \omega)},$$

where $\Omega_\omega = \partial_\omega \Omega$.

Proof. A simple calculation based on (3.2) and (3.4) gives

$$e^{-g(D_\omega \bar{g}_{\lambda \mu} - 2 \bar{S}^\lambda_{\omega \mu} g_{\lambda \alpha})} = D_\omega g_{\lambda \mu} - 2 S^\alpha_{\omega \mu} g_{\lambda \alpha} + \Omega_\omega h_{\lambda \mu} - 2 M_{\omega [\lambda \mu]}$$

$$- 2 N_{(\lambda \mu, \omega)} + \Omega_\omega k_{\lambda \mu} - 2 M_{\omega [\lambda \mu]} - 2 N_{(\lambda \mu, \omega)}.$$

Our assertions follow from above relation by means of (1.4) and (3.1).
THEOREM 3.2. The tensors $a_{\alpha\beta\lambda}$ and $b_{\alpha\beta\lambda}$, defined by

(3.7)a \quad a_{\alpha\beta\lambda} = \Omega_{\alpha\beta} + \Omega_{\beta\alpha} - \Omega_{\lambda\mu\nu}

(3.7)b \quad b_{\alpha\beta\lambda} = \Omega_{\alpha\beta\mu} + \Omega_{\alpha\beta\nu} - \Omega_{\lambda\mu\nu}

may be given by

(3.8)a \quad a_{\alpha\beta\lambda} = 2N_{\alpha\beta\lambda} - 4M_{\alpha\beta\lambda}

(3.8)b \quad b_{\alpha\beta\lambda} = 2M_{\alpha\beta\lambda} + 4N_{\alpha\beta\lambda}

Proof. (3.8) are the results of (3.6), (3.7), and (2.4).

THEOREM 3.3. The equations (3.8) are equivalent to

(3.9)a \quad M_{\alpha\beta\lambda} = \frac{1}{2} H_{\alpha\beta\lambda}

(3.9)b \quad N_{\alpha\beta\lambda} = \frac{1}{2} a_{\alpha\beta\lambda}

where

(3.10) \quad H_{\alpha\beta\lambda} = b_{\alpha\beta\lambda} - 2a_{\alpha\beta\lambda} = 2\Omega_{\alpha\beta\mu}k_{\mu\lambda}

Proof. The last part of (3.10) follows from (3.7). Eliminating $N_{\alpha\beta\lambda}$ from (3.8) and using (2.4) and (2.5), we have

(3.11) \quad 2M_{\alpha\beta\lambda} - 4M_{\alpha\beta\lambda} = H_{\alpha\beta\lambda}

(3.9)a may be obtained from (3.11) by means of (1.3)b and the following relation equivalent to (2.6):

(3.12) \quad 2M_{\alpha\beta\lambda} = -M_{\alpha\beta\lambda}

The second relation (3.9)b may be obtained by substituting (3.9)a for $M_{\alpha\beta\lambda}$ into (3.8)a.

Now that we have found the tensors $M_{\alpha\beta\lambda}$ and $N_{\alpha\beta\lambda}$ in terms of $g_{\alpha\beta\mu}$, it is possible to complete the relation (3.4) as in the following main theorem:

THEOREM 3.4. Under the conformal change (3.2), the Einstein's connections $\tilde{\Gamma}_{\alpha\beta\lambda}$ and $\Gamma_{\alpha\beta\lambda}$ are related as

(3.13) \quad \tilde{\Gamma}_{\alpha\beta\lambda} = \Gamma_{\alpha\beta\lambda} + \frac{1}{2g}(\Omega_{\beta\mu\nu}h^{\beta\nu} + 2\Omega_{\alpha\mu\nu}k_{\mu\nu} - \Omega_{\alpha\beta\nu}h_{\mu\nu})

Proof. Substituting (3.9) into (3.4), we have (3.13).

References

1. V. Hlavaty, Geometry of Einstein's unified field theory, P. Noordhoff Ltd., 1957
2. K.T. Chung and C.H. Cho, Some recurrence relations and Einstein's connection in


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