OSCILLATORY PROPERTIES FOR NONLINEAR SECOND ORDER DIFFERENTIAL EQUATIONS WITH DAMPED TERM

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1. Introduction

Consider the following nonlinear second order differential equation with damped term

\[ y''(t) + p(t)y'(t) + q(t)f(y(t)) = 0 \]

and

\[ y''(t) + p(t)y'(t) + q(t)f(y(g(t))) = 0, \]

where \( p, q \in C[\tau_0, \infty) \), \( f \in C(R) \), \( yf(y) > 0 \) for \( y \neq 0 \), \( g'(t) \geq 0 \) for \( t \geq t_0 \) and \( \lim_{t \to \infty} g(t) = \infty \). We define \( r(t) = \exp\left(\int_{\tau_0}^{t} p(s) \, ds\right) \). We restrict our attention to solutions \( y(t) \) of (1) which exist on some half-line \([\tau_0, \infty)\) and are nontrivial for all large \( t \). A solution \( y(t) \) of (1) is called oscillatory if \( y(t) \) has zeros for arbitrarily large \( t \), otherwise, a solution is said to be nonoscillatory. Equation (1) is oscillatory if all solutions of (1) are oscillatory. Recently, Yeh [8] proved some oscillatory results for equation (1) by using Kamenev's [4] method. Many author's have studied equation (1) (see [1, 3, 5-7]). In this paper, we propose another simple but useful oscillation criterion for equations (1) and (2). Especially, our results can be applied to all examples of Yeh [8], and also to the Emden-Fouler equation and the Fermi-Thomas equation.

2. Oscillation theorems

**THEOREM 1.** Let \( f'(y) \) exist and \( f'(y) > 0 \) for \( y \in R' \equiv R - \{0\} \). If

\[ \int_{\tau_0}^{\infty} r(t)q(t) \, dt = \infty \quad \text{and} \quad \int_{\tau_0}^{\infty} \frac{dt}{r(t)} = \infty, \]

then equation (1) is oscillatory.

**Proof.** Suppose that \( y(t) \) is a nonoscillatory solution of (1). Without loss of generality, we may assume that \( y(t) > 0 \) for \( t \geq t_1 \geq t_0 \), since a parallel argument holds when \( y(t) < 0 \). Multiplying (1) by \( r(t)/f(y(t)) \) and integr-
ating from \( t_1 \) to \( t \), we obtain
\[
\int_{t_1}^{t} \frac{r(s)y''(s)}{f(y(s))} ds + \int_{t_1}^{t} \frac{r(s)p(s)y'(s)}{f(y(s))} ds + \int_{t_1}^{t} r(s)q(s) ds = 0.
\]
By using \( r'(t) = r(t)p(t) \) and (4), we have
\[
\int_{t_1}^{t} \frac{y'(t)r(t)}{f(y(t))} - C + \int_{t_1}^{t} \frac{r(s)f'(y(s)) (y'(s))^2}{(f(y(s)))^2} ds + \int_{t_1}^{t} r(s)q(s) ds = 0,
\]
whence we obtain
\[
\frac{y'(t)r(t)}{f(y(t))} \leq C - \int_{t_1}^{t} r(s)q(s) ds,
\]
where \( C \) is a constant.

By (3) and (6), we can obtain
\[
y'(t) < 0 \quad \text{for} \quad t \geq t_2 \geq t_1.
\]
From (5) with using (3) and (7), it follows that there is \( t_3 \geq t_2 \) such that
\[
3 + \int_{t_3}^{t} \frac{r(s)f'(y(s)) (y'(s))^2}{(f(y(s)))^2} ds \leq \frac{r(t) [-y'(t)]}{f(y(t))}.
\]
Multiplying (8) by
\[
\frac{f'(y(t)) [-y'(t)]}{f(y(t))} \left[ 3 + \int_{t_3}^{t} \frac{r(s)f'(y(s)) (y'(s))^2}{(f(y(s)))^2} ds \right]^{-1} \geq 0
\]
and integrating from \( t_3 \) to \( t \), we have
\[
\log \frac{f(y(t_3))}{f(y(t))} \leq \log \left[ 3 + \int_{t_3}^{t} \frac{r(s)f'(y(s)) (y'(s))^2}{(f(y(s)))^2} ds \right].
\]
By (8) and (9), we obtain
\[
f(y(t_3)) \leq r(t) [-y'(t)] \quad \text{for} \quad t \geq t_3.
\]
Dividing (10) through by \( r(t) \) and integrating from \( t_3 \) to \( t \), we have
\[
y(t) \leq y(t_3) - f(y(t_3)) \int_{t_3}^{t} \frac{1}{r(s)} ds, \quad t \geq t_3,
\]
which, because it is supposed that \( y(t) > 0 \) for \( t \geq t_1 \), contradicts (3).

Q. E. D.

REMARK. Theorem 1 corresponds to Theorem 1 of Yeh [8] and applies to all Examples of Yeh [8].

EXAMPLE 1[8]. Consider the equation
\[
y''(t) + \frac{1}{t} y'(t) + \frac{1}{t^2} y(t) = 0, \quad t \geq 1.
\]
Since \( r(t) = \exp \left( \int_{1}^{1} ds \right) = \exp (\log t) = t \), all conditions of Theorem 1 are satisfied. Hence, equation (11) is oscillatory.

EXAMPLE 2[8]. Consider the equation
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\[ y''(t) + \frac{1}{2t} y'(t) + \frac{1}{4t} y(t) = 0, \quad t \geq 1. \]

Since \( r(t) = \sqrt{t} \), all conditions of Theorem 1 are satisfied. Hence, equation (12) is oscillatory.

**Example 3.** Consider the equation

\[ y''(t) + (\sin t)y'(t) + (1 - \cos t)y(t) = 0, \quad t \geq \frac{\pi}{2}. \]

Since \( r(t) = \exp(\int_{t/2}^{t} \sin s \, ds) = \exp(-\cos t) \), all conditions of Theorem 1 are satisfied. Hence, equation (13) is oscillatory. In fact, \( y(t) = \sin t \) is a solution of (13).

**Remark.** Theorem 1 is easier to apply to Example 3 rather than Theorems 1 and 2 of Yeh [8].

**Theorem 2.** Let \( q(t) \geq 0 \) and \( f(y)/y \geq k > 0 \) for \( y \neq 0 \). If (3) holds, then equation (1) is oscillatory.

**Proof.** Assume that \( y(t) \) is a nonoscillatory solution of (1). Multiplying (1) by \( r(t)/y(t) \) and integrating from \( t_1 \) to \( t \), where \( t_1 \) is so chosen that \( y(t) > 0 \) for \( t > t_1 \), we obtain

\[ \int_{t_1}^{t} \frac{r(s)y''(s)}{y(s)} \, ds + \int_{t_1}^{t} \frac{r(s)p(s)y'(s)}{y(s)} \, ds + \int_{t_1}^{t} kr(s)q(s) \, ds \leq 0. \]

By (14) with using \( r'(t) = r(s)p(s) \), we have

\[ y'(t)r(t) - C + \int_{t_1}^{t} \frac{r(s)(y'(s))^2}{(y(s))^2} \, ds + k \int_{t_1}^{t} r(s)q(s) \, ds \leq 0, \]

where \( C \) is a constant.

By (3) and (15), we obtain

\[ y'(t) < 0 \quad \text{for} \quad t \geq t_2 \geq t_1. \]

From (15) with using (3) and (16), it follows that there is a \( t_3 \geq t_2 \) such that

\[ 3 + \int_{t_3}^{t} \frac{r(s)(y'(s))^2}{(y(s))^2} \, ds \leq \frac{r(t)[ -y'(t)]}{y(t)}. \]

Multiplying (17) by

\[ \frac{[ -y'(t)]}{y(t)} \left[ 3 + \int_{t_3}^{t} \frac{r(s)(y'(s))^2}{(y(s))^2} \, ds \right]^{-1} \]

and integrating from \( t_3 \) to \( t \), we have

\[ \log \frac{y(t_3)}{y(t)} \leq \log \left[ 3 + \int_{t_3}^{t} \frac{r(s)(y'(s))^2}{(y(s))^2} \, ds \right]. \]

By (17) and (18), we obtain

\[ y(t_3) \leq r(t) \left[ -y'(t) \right], \quad t \geq t_3. \]
The rest of the proof proceeds as Theorem 1. Q.E.D.

REMARK. Theorem 2 corresponds to Theorem 2 of Yeh [8] and applies to Examples 1-3.

3. The case with deviating argument

THEOREM 3. Let \( f'(y) \) exist, \( f'(y) > 0 \) for \( y \in \mathbb{R}' = \mathbb{R} - \{0\} \) and \( q(t) \geq 0 \).
If (3) holds, then equation (2) is oscillatory.

Proof. Suppose that \( y(t) \) is a nonoscillatory solution of (2). Without loss of generality, we may assume that \( y(t) > 0 \) for \( t \geq t_1 \geq t_0 \), since a parallel argument holds when \( y(t) < 0 \). Multiplying (2) by \( \frac{r(t)}{f(y(g(t)))} \) and integrating from \( t_2 \) to \( t \), where \( t_2 \) being so large that \( g(t) \geq t_1 \), we obtain

\[
\int_{t_2}^{t} \frac{r(s)y''(s)}{f(y(g(s)))} \, ds + \int_{t_2}^{t} \frac{r(s)p(s)y'(s)}{f(y(g(s)))} \, ds + \int_{t_2}^{t} r(s)q(s) \, ds = 0.
\]

By (19), \( f'(y) > 0 \) and \( g'(t) \geq 0 \), we have

\[
\frac{y'(t)r(t)}{f(y(g(t)))} \leq C - \int_{t_2}^{t} r(s)q(s) \, ds,
\]
where \( C \) is a constant.

From (3) and (20), we obtain

\[
y'(t) < 0 \quad \text{for} \quad t \geq t_3 \geq t_2.
\]

On the other hand, from (2), we may write

\[
r(t)y''(t) + r(t)p(t)y'(t) + r(t)q(t)f(y(g(t))) = 0.
\]

By integrating (22) from \( t_4 (> t_3) \) to \( t \), we have

\[
r(t)y'(t) \leq r(t_4)y'(t_4).
\]

By dividing this and integrating \( t_4 \) to \( t \), we obtain

\[
y(t) - y(t_4) \leq r(t_4)y'(t_4) \int_{t_4}^{t} \frac{ds}{r(s)},
\]
which, (21) and (3) lead a contradiction as \( t \to \infty \). Q.E.D.

EXAMPLE 4. Consider the equation

\[
y''(t) + (\sin t)y'(t) + (1 - \cos t)y(t + 2\pi) = 0, \quad t \geq \frac{\pi}{2}.
\]
Equation (23) is oscillatory by Theorem 3. In fact, \( y(t) = \sin t \) is an oscillatory solution of (23).

4. Application to the Emden–Fowler equation

The Emden–Fowler equation [2, 7] encountered in astrophysics and Fermi–Thomas equation in atomic physics take the following form

\[
(t^{p}y'(t))' + t^{\gamma}y(t) = 0, \quad t \geq 1,
\]
where \( p, \lambda, \gamma \) positive constants, \( \gamma \) the ratio of odd integers. Equation (24)
Oscillatory properties for nonlinear second order differential equations with damped term is written in the form

(25) \[ y''(t) + \frac{P}{t} y'(t) + t^{2-\gamma} y(t)^\gamma = 0. \]

Bobisud [2] proved that equation (25) was oscillatory provided \( \gamma \leq 1 \) and \( \lambda \geq p - 1 \). Wong [7] studied equation (25) for \( p > 1 \). Here, we can show that Theorem 1 covers the other case, for example, \( \gamma = 1 \), \( p = 1 \) and \( \lambda = -(1/2) \). Consider the equation (25) for \( \gamma = 1 \), \( p = 1 \) and \( \lambda = -(1/2) \). Since \( r(t) = \exp(\int_1^t \frac{1}{s} ds) = t \), all conditions of Theorem 1 are satisfied. Hence, equation (25) is oscillatory, and also equation (24) is oscillatory.

References


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