ANGULAR SEPARATIONS OF FINITE SETS IN $E^2$

By E.P. Merkes*

1. Introduction

For an integer $n>1$, let $K_n$ denote a set of $n$ points in the Euclidean plane $E^2$. A partition of $K_n$ is an unordered pair of nonempty subsets $A$ and $B$ of $K_n$ such that $A \cup B = K_n$ and $A \cap B = \emptyset$. We denote such a partition by $[A, B]$ (or $[B, A]$). The number of partitions of $K_n$ is clearly $2^{n-1} - 1$.

For a given real $\theta$, $0 < \theta < \pi$, a partition $[A, B]$ of $K_n$ is called a $\theta$-separation, written $(A, B)$, if there exists two lines that intersect in an angle of measure $\theta$ such that $A$ and $B$ respectively lie in the interior of the opposite vertical angles of measure $\theta$ determined by the pair of lines. We denote by $\eta(\theta, n)$ the maximum number of $\theta$-separations over all sets $K_n$ of $n$ points in $E^2$. In particular, $\eta(\theta, 2) = 1$ for all choices of $\theta$. If $\theta > \frac{\pi}{3}$ and if the points of $K_3$ determine a triangle, each angle of which is less than $\theta$, then there are three $\theta$-separations of $K_3$. Hence, $\eta(\theta, 3) = 3$ for $\theta > \frac{\pi}{3}$ since there are only three partitions of a set of three points in $E^2$.

A few years ago, the author and one of his students proved that $\eta\left(\frac{\pi}{2}, n\right) = n$ for $n > 2$ [1]. The solution of this combinatorial problem was needed to determine the number of distinct domains of univalence for certain families of rational functions. The problem of finding $\eta(\theta, n)$ for other choices of $\theta$ does have implications in the theory of univalent functions although it appears to be an interesting and nontrivial problem itself. The method of proof in [1] can be easily extended to show $\eta(\theta, n) = n$ for $n > 2$ when $\frac{\pi}{3} < \theta \leq \frac{\pi}{2}$. In this paper, we prove $\eta(\theta, n) = n - 1$ when $0 < \theta \leq \frac{\pi}{3}$. When $\theta = \pi$,
the two lines of the $\theta$-separation coincide and $\eta(\pi, n) = \frac{n(n-1)}{2}$. The determination of $\eta(\theta, n)$ for $\frac{\pi}{2} < \theta < \pi$ is an open question.

More explicitly, we prove here the following result.

**Theorem.** For $0 < \theta \leq \frac{\pi}{3}$, there are at most $n-1$ $\theta$-separations of $n$ distinct points in the Euclidean plane. For each $n > 1$ there is a set of $n$ distinct points in the plane such that there are exactly $n-1$ $\theta$-separations.

Our proof is by mathematical induction.

2. Preliminaries

A $\theta$-separation for a set $K_3 = \{k_1, k_2, k_3\}$ of three points in the plane has one point, $k_1$ say, in the interior of an angle of measure $\theta$ whereas the other two points are in the interior of the opposite vertical angle. Therefore the angle $\angle k_2 k_1 k_3$ at $k_1$ must have measure less than $\theta$. If $0 < \theta \leq \frac{\pi}{3}$, it follows that at least one of the other two angles in the triangle determined by $k_1, k_2,$ and $k_3$ must exceed $\frac{\pi}{3}$ in measure. This implies that the vertex of this angle is a point that cannot be separated from the other two vertices by a $\theta$-separation. We conclude $\eta(\theta, 3) \leq 2$. By selecting the points $k_1, k_2,$ and $k_3$ such that two of the angles of the triangle determined by these three points each have measure less than $\theta$, we prove $\eta(\theta, 3) \geq 2$ and, hence, $\eta(\theta, 3) = 2$ when $0 < \theta \leq \frac{\pi}{3}$. This can serve as the starting point of our induction.

Suppose $[A, B]$ is a partition of the set $K_n$ of $n$ points in the plane and $k \notin K_n$. The set $K_{n+1} = K_n \cup k$ (actually $K_n \cup \{k\}$ but the braces are dropped for simplicity of notation) has two partitions that naturally correspond to the partition $[A, B]$ of $K_n$, namely, $[A \cup k, B]$ and $[A, B \cup k]$. If either is a $\theta$-separation of $K_{n+1}$, then by deleting the point $k$ we conclude that the partition $[A, B]$ was a $\theta$-separation of $K_n$. Hence, a $\theta$-separation $(A, B)$ of $K_n$ corresponds to at most two $\theta$-separations, $(A \cup k, B)$ and $(A, B \cup k)$, of $K_{n+1}$. The only other type of $\theta$-separation of $K_{n+1}$ that can arise is $(k, K_n)$.

**Lemma 1.** Let $0 < \theta \leq \frac{\pi}{3}$. If $(k, K_n)$ is a $\theta$-separation of $K_{n+1} = K_n \cup k$, $k \notin K_n$, then for any $\theta$-separation $(A, B)$ of $K_n$ at most one of the two par-
Angular separations of finite sets in $E^2$

Proof. Let $a \in A$ and $b \in B$, where $(A, B)$ is a $\theta$-separation of $K_n$. Since $(k, K_n)$ is a $\theta$-separation of $K_{n+1}$, the triangle determined by $a$, $b$, and $k$ has an angle of measure less than $\theta$ at $k$. This implies the angle at $a$ or at $b$ of this triangle has measure exceeding $\theta$ and, hence, $a$ or $b$ cannot be separated by a $\theta$-separation from the other two vertices of the triangle. We conclude that either $[A, B \cup k]$ or $[A \cup k, B]$ is not a $\theta$-separation of $K_{n+1}$.

The next lemma is stated in a more general form than is necessary for the proof of our theorem. It is because this lemma cannot be further extended to the case when $\frac{\pi}{2} < \theta < \pi$ that the methods of proof in this paper and in [1] fail for choices of $\theta$ beyond $\frac{\pi}{2}$. In the lemma, we use the notation $E-D$ for the complement of the set $D$ in $E$, that is, for the set of all points of $E$ that are not points of $D$.

**Lemma 2.** Let $0 < \theta \leq \frac{\pi}{2}$. If $(A, B)$ is a $\theta$-separation of $K_n$ and if $A_1$, $B_1$ are respectively nonempty proper subsets of $A$ and $B$, then the partition $[A_1 \cup B_1, K_n - (A_1 \cup B_1)]$ is not a $\theta$-separation of $K_n$.

**Proof.** Select a coordinate system such that one line of a $\theta$-separation $(A, B)$ of $K_n$ is the horizontal (real) axis, the other line is in the first and third quadrants (or the vertical axis if $\theta = \frac{\pi}{2}$), and the origin is at the point of intersection of these lines. Then the only points in the first and third quadrant that are on a line which separates $A_1 \cup B_1$ and $K_n - (A_1 \cup B_1)$ into opposite half-planes must be points in the interior of the vertical angles of measure $\theta$ of the $\theta$-separation. The angle between two such lines, therefore, has measure less than $\theta$. Hence, the partition $[A_1 \cup B_1, K_n - (A_1 \cup B_1)]$ cannot be a $\theta$-separation of $K_n$.

**Lemma 3.** Let $0 < \theta \leq \frac{\pi}{2}$. If $k \in K_n$, then there is at most one partition $[A, B]$ of $K_n$ such that both $(A \cup k, B)$ and $(A, B \cup k)$ are $\theta$-separations of $K_n \cup k$.

**Proof.** If $[A, B]$ is a partition of $K_n$, then each other partition of $K_n$ must have one of the following forms:

$[A_1, K_n - A_1]$, $[B_1, K_n - B_1]$, $[A_1 \cup B_1, K_n - (A_1 \cup B_1)]$,

where $A_1$, $B_1$ are respectively proper nonempty subsets of $A$ and $B$. Suppose
(A, B∪k) and (A∪k, B) are \( \theta \)-separations of \( K_{n+1} = K_n ∪ k \). Hence, \( (A, B) \) is a \( \theta \)-separation of \( K_n \). Now \([A_1∪B_1, K_n−(A_1∪B_1)]\) is not a \( \theta \)-separation of \( K_n \) by Lemma 2. Therefore, adjoining the point \( k \) to either of the sets in this partition cannot lead to a \( \theta \)-separation of \( K_{n+1} \). Since \( (A, B∪k) \) is a \( \theta \)-separation of \( K_{n+1} \), the partition \([A_1∪k, K_n−A_1]\) cannot by Lemma 2 be a \( \theta \)-separation of \( K_{n+1} \). Indeed, points from the first set \( A \) of the \( \theta \)-separation \( (A, B∪k) \) of \( K_{n+1} \) are transferred to the second set while a point of the second set, \( k \), is transferred to the first set in building the partition \([A_1∪k, K_n−A_1]\). Lemma 2 assures us that such a transformation does not produce \( \theta \)-separations. Similarly \([A_1∪k, (K_n−A_1)∪k]\) is not a \( \theta \)-separation of \( K_{n+1} \) since \( (A∪k, B) \) is a \( \theta \)-separation. By symmetry what has been proved for \( A \) also applies when \( A \) is replaced by \( B \). Thus, there is no second partition \([A, B]\) such that \( (A∪k, B) \) and \( (A, B∪k) \) are \( \theta \)-separations of \( K_{n+1} \).

3. Proof of the Theorem.

Assume for some integer \( n \geq 3 \) that \( \eta(\theta, n) \leq n−1 \), where \( 0<\theta \leq \frac{\pi}{3} \). Let \( K_{n+1} \) be a set of \( n+1 \) points in the plane and let \( k \in K_{n+1}, K_n = K_{n+1}−k \). The number of \( \theta \)-separations of \( K_n \) is at most \( n−1 \). Each \( \theta \)-separation of \( K_{n+1} \), except \((k, K_n)\) if it is a \( \theta \)-separation, arises from the partitions \([A∪k, B]\) or \([A, B∪k]\), where \( (A, B) \) is a \( \theta \)-separation of \( K_n \). If \((k, K_n)\) is a \( \theta \)-separation of \( K_{n+1} \), then by Lemma 1 at least one of the partitions \([A∪k, B]\) or \([A, B∪k]\) is not a \( \theta \)-separation of \( K_{n+1} \). Hence, the number of \( \theta \)-separations of \( K_{n+1} \) is at most one greater than the number of \( \theta \)-separations of \( K_n \) in this case. On the other hand, if \([k, K_n]\) is not a \( \theta \)-separation of \( K_{n+1} \), then there is at most one \( \theta \)-separation, \( (A, B) \) say, of \( K_n \) such that both \((A∪k, B)\) and \((A, B∪k)\) are \( \theta \)-separations of \( K_{n+1} \) by Lemma 3. Again the number of \( \theta \)-separations of \( K_{n+1} \) is at most one greater than those of \( K_n \). It follows that \( \eta(\theta, n+1) \leq n \). Since \( \eta(\theta, 3) = 2 \), we have by induction \( \eta(\theta, m) \leq m−1 \) for all integers \( m \geq 3 \). (The inequality is also trivially true for \( m=2 \).)

It remains to prove \( \eta(\theta, m) = m−1 \) when \( 0<\theta \leq \frac{\pi}{3} \). This is accomplished by noting that the number of \( \theta \)-separations of \( m \) points on a line is exactly \( m−1 \) for \( m \geq 2 \).

Remark. If \( \frac{\pi}{3}<\theta \leq \frac{\pi}{2} \), the proof in [1] can easily be extended to establish the inequality \( \eta(\theta, m) \leq m \) for \( m > 2 \). To prove equality can hold, we determine the set \( K_m \) as follows. Select \( m−2 \) points in a coordinate plane...
Angular separations of finite sets in $E^2$

of the form $(x, 0)$, where $x$ is in the open interval $\cot \frac{\theta}{2} < x < \tan \theta$. The remaining two points are $(0, 1)$ and $(0, -1)$. The number of $\theta$-separations of $K_m$ in this case is exactly $m$.

4. Open Questions.

We have already mentioned that the value of $\eta(\theta, n)$ for $\frac{\pi}{2} < \theta < \pi$ is unknown. Of course, for $n > 2$ we have $n \leq \eta(\theta, n) \leq \frac{n(n-1)}{2}$. We suspect the value of $\eta(\theta, n)$ for sufficiently large $n$ changes at each $\theta$ of the form $\frac{(m-2)\pi}{m}$ $(m=3, 4, 5, \ldots)$, the measure of the angles of a regular polygon of $m$ sides.

The beauty of the problem so far is that its resolution required only the most elementary mathematics. However, is there a shorter proof of the known results perhaps using techniques from the subject of "convexity"?

Finally, are there analogues of even the known results in Euclidean space $E^d, d > 2$? Since we know of no application for this generalization, we have not attempted an extension to higher dimensions. Nonetheless the problem does appear to be of interest.

Reference


Department of Mathematical Sciences
University of Cincinnati
Cincinnati, Ohio 45221
U. S. A.