0. Introduction

Recently the present authors [4] studied infinitesimal variations of invariant submanifolds with normal \((f, g, u, v, \lambda)\)-structure. Yano and Kon [3] studied infinitesimal variations of an even dimensional sphere.

The purpose of the present paper is to study infinitesimal variations of submanifolds of \(S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})\).

In §1, we state some of known results on structures [1] which \(S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})\) admits.

In §2, we investigate infinitesimal variations of various kinds of submanifolds.

In §3, we study infinitesimal variations of \(f\)-invariant submanifold and \(k\)-invariant one.

§4 is devoted to the study of isometric variation and \(f\)-preserving variation of a \(f\)-invariant and \(k\)-invariant submanifold.

And in §5, on a compact \(f\)-invariant and \(k\)-invariant submanifold with induced \((f, g, u, v, \lambda)\)-structure, we investigate some variation-preserving relations.

The last §6 is devoted to the study of infinitesimal variations of \(f\)-anti-invariant submanifold and \(k\)-anti-invariant one.

1. Preliminaries

Let \(S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})\) be a submanifold of codimension 2 of \((2n + 2)\) dimensional Euclidean space \(E^{2n+2}\) and be covered by a system of coordinate neighbourhoods \(\{U, x^h\}\), where here and in the sequel the indices \(h, i, j, k, \ldots\) run over the range \(1, 2, 3, \ldots, 2n\). \((S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})\) is the direct product differentiable manifold of two spheres \(S^n(1/\sqrt{2})\) with radius \(1/\sqrt{2}\) and with its center at the origin in \(E^{n+1}\).)

Let denote by \(Z, C\) and \(D\) the position vector of a point of \(S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})\), the first unit normal vector in the direction opposite to that of

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the radius vector of $S^{2n+1}(1)$ and the second unit normal vector in the direction normal to $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ and tangent to $S^{2n+1}(1)$ respectively.

In $E^{2n+2}$, there exists a natural Kählerian structure

\begin{equation}
F = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix},
\end{equation}

$E$ being the unit square matrix of order $n+1$. Of course, $F$ satisfies

\begin{equation}
F^2 = -I, \quad FU \cdot FV = U \cdot V
\end{equation}

for arbitrary vectors $U$ and $V$ in $E^{2n+2}$, $I$ and $\cdot$ denoting the identity transformation in $E^{2n+2}$ and the inner product of two vectors in Euclidean space respectively.

Now we put

\begin{equation}
Z_i = \partial Z/\partial x^i, \quad g_{ji} = Z_j \cdot Z_i,
\end{equation}

where $Z_i$ are $2n$ linearly independent vectors of $E^{2n+2}$ tangent to $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$.

Applying $F$ to $Z_i$, $C$ and $D$ gives

\begin{equation}
FZ_i = f^k_i Z_k + u_i C + v_i D,
\end{equation}

\begin{equation}
FC = -u^i Z_i + \lambda D,
\end{equation}

\begin{equation}
FD = -v^i Z_i - \lambda C,
\end{equation}

where $f^k_i$ are the components of a tensor field of type $(1, 1)$, $u_i$ and $v_i$ are the components of 1–forms, $\lambda$ is a function on $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$, $u_i$ and $v_i$ are respectively given by $u^i = u_j g^{ji}$ and $v^i = v_j g^{ji}$, $g^{ji}$ being contravariant components of the metric tensor $g_{ji}$.

From (1.2), (1.4), (1.5) and (1.6), we find

\begin{equation}
\begin{cases}
  f_j f^k_i = -\delta^k_j + u_j u^k + v_j v^k, \\
  u_i f^i_j = \lambda v_j, \quad f^i_i u^i = -\lambda v^i, \\
  v_i f^i_j = -\lambda u_j, \quad f^i_i v^i = \lambda u^i, \\
  u_i u^i = v_i v^i = 1 - \lambda^2, \quad u_i v^i = 0, \\
  f^m_i f^i_j g_{ml} = g_{ji} - u_j u_i - v_j v_i.
\end{cases}
\end{equation}

A set of $f, g, u, v$ and $\lambda$ satisfying these equations is called an $(f, g, u, v, \lambda)$–structure. It is verified that $f_{ji} = f^j_i g_{li}$ is skew–symmetric in $j$ and $i$.

Now applying the operator $\nabla_j$ of covariant differentiation with respect to the Riemannian connection to (1.4), (1.5) and (1.6), and taking account
of $V_jk^i=0$, we find

$$
\begin{align*}
V_jf^h_i &= -g_{ji}u^h + \delta_j^h u_i - k_{ji}v^h + k^h v_i , \\
V_ju^i &= f_{ji} - \lambda k_{ji} , \\
V_jv^i &= - k_{ji} f^i + \lambda g_{ji} , \\
V_j\lambda &= - 2v_j ,
\end{align*}
$$

(1.8)

where $k_{ji}$ are the components of the second fundamental tensors with respect to the second unit normal $D$. From (1.7) and the last equation of (1.8), $\lambda$ does not vanish almost everywhere in $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$. Moreover $k_j^h$ are given by the following form [1]:

$$
(k_j^h) = \begin{bmatrix} E & 0 \\ 0 & -E \end{bmatrix},
$$

(1.9)

where $E$ being the unit square matrix of order $n$ and $k_j^h = k_{ji}g^{ih}$. From (1.4) and (1.9), we find [1]

$$
k_j^ju^j = -v_j , \quad k_j^jv^j = -u^j .
$$

(1.10)

We have from the last two equations of (1.8)

$$
k_m^h f_i^m + f_i^h k_i^m = 0 ,
$$

(1.11)

that is, $k_m^h$ and $f_i^m$ are anticommutate with each other.

Let $M^m$ be an $m$-dimensional Riemannian manifold covered by a system of coordinate neighbourhoods $\{V, y^a\}$ and isometrically immersed in $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ by the immersion $i: M^m \rightarrow S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$, where here and in the sequel the indices $a, b, c, \ldots$ run over the range $[1, 2, 3, \ldots, m]$. We identify $i(M^m)$ with $M^m$ and represent the local expression of the immersion $i$ by $x^h = x^h(y^a)$. If we put $B_i^h = \partial_i x^h$, $(\partial_i = \partial / \partial y^i)$, then $B_i^h$ are $m$ linearly independent vectors of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ tangent to $M^m$. Denoting by $g_{cb}$ the Riemannian metric tensor of $M^m$, we have $g_{cb} = g_{ji} B_i^c B_j^b$ since the immersion is isometric. We denote by $C_i^h$ the $2n-m$ mutually orthogonal unit normals to $M^m$, where here and in the sequel the indices $x, y, z$ run over the range $[m+1, m+2, m+3, \ldots, 2n]$.

The transforms $f_i^h B_i^j$ and $k_i^h B_i^j$ of $B_i^j$ by $f_i^h$ and $k_i^h$ are written in the form respectively

$$
(1.12) \quad f_i^h B_i^j = f_j^a B_a^h - f_j^x C_x^h ,
$$

$$
(1.13) \quad k_i^h B_i^j = k_j^a B_a^h + k_j^x C_x^h .
$$

and the transforms $f_i^h C_i^j$ and $k_i^h C_i^j$ of $C_i^j$ by $f_i^h$ and $k_i^h$ in the form

$$
(1.14) \quad f_i^h C_i^j = f_j^a B_a^h + f_j^x C_x^h ,
$$
(1.15) \[ k_i^h C_j^i = k_j^s B_a^h + k_j^s C_x^h. \]

From (1.12) \(\sim\) (1.15), we have

\[
\begin{align*}
 f_{ba} &= -f_{ab}, \quad f_{bx} = f_{xb}, \quad f_{xy} = -f_{yx}, \\
 k_{ba} &= k_{ab}, \quad k_{ax} = k_{xa}, \quad k_{xy} = k_{yx},
\end{align*}
\]

where \( f_{by} = f_{b}^s g_{zy} \), \( f_{yb} = f_{y}^s g_{cb} \), \( k_{by} = k_{b}^s g_{zy} \), and \( k_{yb} = k_{y}^s g_{cb} \). \( g_{zy} \) being the metric tensor of the normal bundle of \( M^m \). From (1.11) \(\sim\) (1.15), we have

\[
\begin{align*}
 f_{b}^s k_{s}^c + k_{s}^a f_{a}^s &= f_{b}^s k_{j}^c - k_{j}^y f_{j}^c, \\
 f_{b}^s k_{s}^x - f_{j}^y k_{j}^x &= k_{s}^a f_{a}^x - k_{j}^y f_{j}^x, \\
 f_{a}^s k_{a}^y + f_{x}^z k_{x}^y &= k_{s}^a f_{a}^y - k_{z}^y f_{z}^y.
\end{align*}
\]

We put

\[ u^h = B_a^h u^a + C_x^h u^x, \quad v^h = B_a^h v^a + C_x^h v^x. \]

We get from (1.10), (1.13), (1.15) and (1.20)

\[
\begin{align*}
 u^a &= -v^b k_a^b - v^x k_a^x, \quad v^a &= -u^b k_a^b - u^x k_a^x, \\
 u^x &= -v^b k_a^b - v^x k_a^x, \quad v^x &= -u^b k_a^b - u^x k_a^x.
\end{align*}
\]

When \( f_b^i B_b^i \) and \( k_b^i B_b^i \) are always tangent to \( M^m \) respectively, that is, when \( f_b^a = 0 \) and \( k_b^a = 0 \), \( M^m \) is said to be \( f \)-invariant and \( k \)-invariant respectively. In order for \( M^m \) to be \( f \)-invariant and \( k \)-invariant respectively, it is necessary and sufficient that

\[
\begin{align*}
 f_{bx} &= f_{y}^s g_{yx} = -f_{j1} B_b^j C_x^i = 0, \\
 k_{bx} &= k_{y}^s g_{yx} = k_{j1} B_b^j B_a^i = 0.
\end{align*}
\]

respectively.

When \( f_b^i B_b^i \) and \( k_b^i B_b^i \) are always normal to \( M^m \) respectively, that is, \( f_b^a = 0 \) and \( k_b^a = 0 \), \( M^m \) is said to be \( f \)-antiinvariant and \( k \)-antiinvariant respectively. In order for \( M^m \) to be \( f \)-antiinvariant and \( k \)-antiinvariant respectively, it is necessary and sufficient that

\[
\begin{align*}
 f_{ba} &= f_{b}^s g_{ca} = f_{j1} B_b^j B_a^i = 0, \\
 k_{ba} &= k_{b}^s g_{ca} = k_{j1} B_b^j B_a^i = 0
\end{align*}
\]

respectively.

Equations of Gauss and those of Weingarten for \( M^m \) are respectively written as

\[ V_c B_b^h = h_{cb}^s C_x^h \]
2. Infinitesimal variations of submanifolds of \( S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2}) \)

We now consider an infinitesimal variation

\[
\bar{x}^h = x^h + \xi^h(y) \epsilon
\]

of a submanifold \( M^m \) of \( S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2}) \), where \( \xi^h \) is a vector field of \( S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2}) \) defined along \( M^m \) and \( \epsilon \) is an infinitesimal. We then have \( \bar{B}_b^h = B_b^h + \partial_i \xi^h \epsilon_i \), where \( \bar{B}_b^h = \bar{B}_b^h + (\nabla_b \xi^h) \epsilon \), neglecting terms of order higher than one with respect to \( \epsilon \). In the sequel, we neglect always terms of order higher than one with respect to \( \epsilon \). Thus putting \( \delta B_b^h = \bar{B}_b^h - B_b^h \), we obtain

\[
(2.2) \quad \delta B_b^h = (\nabla_b \xi^h) \epsilon.
\]

On the other hand, putting

\[
(2.3) \quad \xi^h = \xi^a B_a^h + \xi^z C_x^h,
\]

we have

\[
(2.4) \quad \nabla_b \xi^h = (\nabla_b \xi^a - h_b^a \xi^z) B_a^h + (\nabla_b \xi^z + h_b^a \xi^a) C_x^h.
\]

When the tangent space at a point \( (x^h) \) of the submanifold and that at the corresponding \( (\bar{x}^h) \) of the varied submanifold are parallel, the variation is said to be parallel.

From (2.2) and (2.4) we see that in order for an infinitesimal variation to be parallel, it is necessary and sufficient that

\[
(2.5) \quad \nabla_b \xi^z + h_b^a \xi^a = 0.
\]

We denote by \( \bar{C}_y^h \) mutually orthogonal unit normals to the varied submanifold and by \( \bar{C}_y^h \) vectors obtained from \( \bar{C}_y^h \) by parallel displacement of \( \bar{C}_y^h \) from the varied point \( (\bar{x}^h) \) back to the original point \( (x^h) \). Putting \( \delta C_y^h = \bar{C}_y^h - C_y^h \), we find [2]

\[
(2.6) \quad \bar{C}_y^h = C_y^h - I^h j_i \xi^i C_y^i \xi + \delta C_y^h,
\]
where \( \Gamma^i_{jl} \) are Christoffel symbols formed with \( g_{ji} \). Assuming that \( \partial C^h_j \) are infinitesimals of order one with respect to \( \varepsilon \) and putting
\[
\partial C^h_j = (\eta_j^a B_a^h + \eta_j^x C_x^h) \varepsilon,
\]
we have
\[
\eta_j^a = - (V^a \xi_j + h_b^a \xi_j^b),
\]
where \( \xi_j = \xi_j^x g_{jx} \). The \( \eta_j^x \) appearing in (2.7) is a tensor field of the normal bundle of \( M^m \) satisfying \( \eta_{yx} + \eta_{xy} = 0 \), \( \eta_{yx} \) being defined by \( \eta_{yx} = \eta_j^x g_{xj} \).

(1) Infinitesimal variations of submanifolds of \( S^a(1/\sqrt{2}) \times S^a(1/\sqrt{2}) \) tangent to \( u^h \).

Suppose that a submanifold \( M^m \) of \( S^a(1/\sqrt{2}) \times S^a(1/\sqrt{2}) \) is tangent to \( u^h \). Then we have equation of the form \( B_a^h u^a = u^h \), from which, differentiating covariantly along \( M^m \) and using (1.8), (1.12) and (1.13), we find
\[
h_{cb} \varepsilon u^h C_x^h + B_a^h (V^a u^a) = (f_c^a - \lambda k_c^a) B_a^h - (f_c^x + \lambda k_c^x) C_x^h,
\]
and consequently
\[
(2.9)
\]
In order that the varied submanifold is also tangent to \( u^h(\bar{x}) \), it is necessary and sufficient that we have equation of the form \( B_a^h \bar{u}^a = u^h(\bar{x}) \), from which, putting \( \bar{u}^a = u^a + \delta u^a \),
\[
(B_b^h + \partial \bar{\xi}^b \varepsilon) (u^b + \delta u^b) = u^h + \bar{\xi}^j \partial_j u^h \varepsilon.
\]
Thus using (1.8), we find
\[
(2.10)
\]
On the other hand, using (1.12)~(1.15) and (2.3), we find
\[
(f_c^b \xi_j^i = (f_b^a \xi_j^b + f_j^a \xi^a) B_a^h + (-f_b^a \xi_j^b + f_j^x \xi^x) C_x^h,
\]
\[
\xi_j^i = (f_b^a \xi_j^b + f_j^x \xi^x) B_a^h + (-f_b^a \xi_j^b + f_j^x \xi^x) C_x^h.
\]
Thus, substituting (2.4) and (2.11) into (2.10), we find
\[
B_a^h \delta u^a + [(V^a \xi_j^h + h_b^a \xi_j^b) u^b - (f_b^a \xi_j^b + f_j^x \xi^x) u^b - \lambda (\xi_d^b h_b^a + \xi_j^b k_a^i)] B_a^h \xi
\]
\[
+ [(V^a \xi_j^h + h_b^a \xi_j^b) u^b - (f_b^a \xi_j^b + f_j^x \xi^x)] C_x^h \xi = 0,
\]
or, using (2.9),
Infinitesimal variations of submanifolds of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ \[ B_a^b \delta u^a + (u^b \Gamma^c_{ba} - \xi^c \dot{u}^a + 2\lambda \xi^c k_x^a) B_a^b \xi \]

\[ + (u^b \Gamma^c_{ba} - f_y^c \xi^a + \lambda \xi^c k_x^c) C_x^b \xi = 0, \]

from which

\[ (2.12) \quad \delta u^a = (\mathcal{L} u^a - 2\lambda \xi^a k_x^a) \xi, \]

$\mathcal{L}$ denoting the Lie derivative with respect to $\xi^a$ and

\[ (2.13) \quad u^b \Gamma^c_{ba} - f_y^c \xi^a + \lambda \xi^c k_x^c = 0. \]

Thus we have

PROPOSITION 2.1. In order for an infinitesimal variation (2.1) to carry a submanifold $M_m$ of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ tangent to $u^b$ into a submanifold $\bar{M}_m$ also tangent to $u^b(\bar{x})$, it is necessary and sufficient that (2.13) holds, the variation of $u^a$ being given by (2.12).

(2) Infinitesimal variations of a submanifolds of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ normal to $u^b$.

Suppose that a submanifold $M_m$ of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ is normal to $u^b$. Then we have $B_b^i u_i = 0$, from which differentiating covariantly along $M_m$ and using (1.8), (1.12), (1.13) and (1.20), we find

\[ h_{cb} \xi^a u_x + B_b^i (f^e a B_{ai} - f^e C_{xi}) - \lambda k_{eb} = 0, \]

where $B_{ai} = B_a^j g_{ji}$ and $C_{xi} = C_x^j g_{ji}$ and consequently

\[ h_{cb} \xi^a u_x + f_{eb} - \lambda k_{eb} = 0. \]

Thus, $h_{cb} \xi^a u_x$ and $\lambda k_{eb}$ being symmetric and $f_{eb}$ being skew symmetric in $c$ and $b$, we have

\[ (2.14) \quad h_{cb} \xi^a u_x = \lambda k_{eb}, \quad f_{eb} = 0. \]

Thus we have

PROPOSITION 2.2. If a submanifold $M_m$ of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ is normal to $u^b$, then $M_m$ is $\xi$-antinvariant.

Now in order that the varied submanifold is also normal to $u^b(\bar{x})$, it is necessary and sufficient that we have $B_b^i u_i (\bar{x}) = 0$, from which

\[ [B_b^i + (\partial_0 \xi^i) \bar{x}] [u_i + \xi^j \partial_0 u_i \xi] = 0. \]

Thus, using (1.8), we find

\[ (\Gamma^i_{cb} \xi^i) u_i + f_{ji} \xi^j B_b^i - \lambda k_{ji} B_b^i \xi = 0. \]

Substituting (2.4) and (2.11) into this equation, we have
from which, using (2.14),

\[(V_{b \xi} u + f_{y b} \xi^x - \lambda^{\xi a} k_{a b} - \lambda^{\xi x} k_x^b) = 0.\]

Thus we have

PROPOSITION 2.3. In order for an infinitesimal variation \(2.1\) to carry a submanifold \(M^m\) of \(S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})\) normal to \(u^k\) into a submanifold normal to \(u^k(x)\), it is necessary and sufficient that \((2.15)\) holds.

(3) Infinitesimal variations of submanifolds of \(S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})\) tangent to \(v^h\).

Suppose that a submanifold \(M^m\) of \(S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})\) is tangent to \(v^h\). Then we have equation of the form \(B_a^{h a} = \nu^h\), from which, differentiating covariantly along \(M^m\) and using \((1.8)\) and \((1.12)\sim(1.14)\), we have

\[
\begin{align*}
(V_{\nu^h}) u &= k_{e a} f_{e a} + k_{x a} f_{x a} + \lambda \delta_{a}, \\
h_{e a} \xi^a &= -k_{e a} f_{e a} + k_{x a} f_{x a}.
\end{align*}
\]

Now, in order that the varied submanifold is also tangent to \(v^h(x)\), it is necessary and sufficient that we have equation of the form \(B_a^{h a} = \nu^h(x)\), from which, putting \(\bar{\nu}^a = \nu^a + \delta \nu^a\),

\[
(B_{b \xi} + \partial_{b \xi} k_{j j}) \ (v^h + \partial \nu^h) = v^h + \xi^i (\partial_i v^h).\]

Thus, using \((1.8)\), we have

\[
B_a^{h a} \partial \nu^a + \{ (V_{b \xi} v^b) - \lambda^{\xi b} + \xi^j k_{j j} f^{h i} \} \varepsilon = 0.
\]

Substituting (2.3), (2.4), (1.12), (1.14) and (2.11) into this equation, we find

\[
B_a^{h a} \partial \nu^a + \{ (V_{b \xi} \xi^a - h_{b a} \xi^x) v^b - \lambda^{\xi b} - (\xi^a k_{a b} + \xi^x k_x^b) f^a_b \\
- (\xi^x k_x^a + \xi^y k_y^a) f^a_x \} B_a^{h e} \\
+ \{ (V_{b \xi} \xi^a + h_{b a} \xi^x) v^b - \lambda^{\xi b} + (\xi^a k_{a b} + \xi^y k_y^b) f^a_x \\
- (\xi^x k_x^a + \xi^y k_y^a) f^a_x \} C_a^{h e} = 0,
\]

from which, using (2.16) and (1.18),

\[
\partial \nu^a = \{ \partial \nu^a + 2 (k_{a b} f_{b a} + k_{x a} f_{x a}) \xi^x \} e,
\]

\(\mathcal{L}\) denoting the Lie derivative with respect to \(\xi^a\) and

\[
\nu^b V_{b \xi} \xi^x - \lambda^{\xi x} + (k_{a b} f_{a x} - k_{x a} f_{x a}) \xi^y = 0.
\]

Thus we have
PROPOSITION 2.4. In order for an infinitesimal variation (2.1) to carry a submanifold $M^m$ of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ tangent to $v^h$ into a submanifold tangent to $v^h(\bar{x})$, it is necessary and sufficient that (2.18) holds, the variation of $v^a$ being given by (2.17).

(4) Infinitesimal variations of submanifolds of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ normal to $v^h$.

Suppose that a submanifold $M^m$ of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ is normal to $v^h$. Then we have equation of the form $B_b^i v_i = 0$ or $B_b^i\lambda = 0$, which shows that $\lambda$ is constant along $M^m$. Differentiating $B_b^i v_i = 0$ covariantly along $M^m$ and using (1.8), (1.12) and (1.13), we find

\begin{equation}
(2.19) \quad h_{cb}^x v_x + \lambda g_{cb} = f_b^a k_{ac} + f_b^\alpha k_{x}^\alpha = 0.
\end{equation}

Now, in order that the varied submanifold is also normal to $v^h(\bar{x})$, it is necessary and sufficient that we have $\bar{B}_b^i v_i(\bar{x}) = 0$, from which

\begin{equation}
(B_b^i + (\partial b^i)_c) (v_i + (\xi c \partial v_i)c) = 0,
\end{equation}

Thus using (1.8), (1.12), (2.3) and (2.11), we have

\begin{equation}
(\bar{V} b^i c) v_i + f_{cb} (\xi b k_{a}^c + \xi x^c k_x^a) + f_{cb} (\xi a k_{x}^c + \xi y k_x^y) + \lambda \xi_b = 0.
\end{equation}

Therefore substituting (2.4) into this equation, we have

\begin{equation}
(\bar{V} b^i c + h_b c^x k_{x}^c) v_x + f_{cb} (\xi b k_{a}^c + \xi x^c k_x^a) + f_{cb} (\xi a k_{x}^c + \xi y k_x^y) + \lambda \xi_b = 0,
\end{equation}

or, using (2.19)

\begin{equation}
(\bar{V} b^i c) v_x + f_{cb} (\xi b k_{x}^c + \xi x^c k_x^x) + f_{cb} (\xi a k_{y}^c + \xi y k_y^c) + \lambda \xi_b = 0.
\end{equation}

Thus we have

PROPOSITION 2.5. In order for an infinitesimal variation (2.1) to carry a submanifold $M^m$ of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ normal to $v^h$ into a submanifold normal to $v^h(\bar{x})$, it is necessary and sufficient that (2.20) holds.

When $\xi x = 0$, that is, the variation vector $\xi^h$ is tangent to the submanifold, we say that the variation is tangential and when the variation vector $\xi^h$ is normal to the submanifold, that is, $\xi^a = 0$, we say that the variation is normal.

3. Infinitesimal variations of $f$–invariant submanifold and $k$–invariant one of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$.

We assume that $M^m$ is an $f$–invariant submanifold of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$. We then have
Differentiating the first equation of (3.1) covariantly along $M^m$ and using (1.8), (1.13) and (3.1), we find

\[
(-g_{cb}u^a + u_0 \delta^a_e - k_c v^a + v_b k^a_e)B_a^h +
\]

\[
( -g_{cb}u^a - k_c v^a + v_b k^a_e + h_c f_b^a)C_x^h
\]

\[
= (F_c f_b^a)B_a^h + f_b^a h_c u^a C_x^h,
\]

and consequently, comparing the tangential and normal parts,

(3.2) \[
V_c f_b^a = -g_{cb}u^a + u_0 \delta^a_e - k_c v^a + v_b k^a_e
\]

and

(3.3) \[
h_c f_b^a - h_e f_b^e = g_{cb}u^a + k_c v^a - v_b k^a_e,
\]

from which, taking the skew symmetric part,

(3.4) \[
\]

Differentiating the second equation of (3.1) covariantly along $M^m$ and using (1.8), (1.13), (1.25) and (3.1), we find

\[
( u_0 \delta^a_e - k_c v^a + k^a d v - h_b f_b^e)B_a^h + (v_y k^a_x - k_c v^a)C_x^h
\]

\[
= (V_c f_y^a)C_x^h - h_e a f_y^a B_a^h,
\]

and consequently

(3.5) \[
\]

which is equivalent to (3.3) and

\[
(3.6) \]

We now consider an infinitesimal variation (2.1) and assume that it carries the $f$-invariant submanifold $M^m$ of $S^a(1/\sqrt{2}) \times S^a(1/\sqrt{2})$ into an $f$-invariant submanifold. Then we have

\[
f_i^h(x + \xi \varepsilon)B_b^i = (f_b^a + \delta f_b^a)B_a^h,
\]

that is,

\[
(f_i^h + \xi \varepsilon \partial_j f_i^h) - (B_b^i + \partial_b^i \xi \varepsilon) = (f_b^a + \delta f_b^a)B_a^h + \partial_a \xi \varepsilon,
\]

from which, using (1.8), we obtain

\[
[f_i^h + \xi \varepsilon (-\Gamma_j^i f_i^h + \Gamma_j^i \varepsilon + h_i^a u^h + \partial_j^h u_i
\]

\[-k_j^h u_i + k_j^h v_i) \xi \varepsilon] (B_b^i + \partial_b^i \xi \varepsilon)
Infinitesimal variations of submanifolds of $S^*\left(1/\sqrt{2}\right) \times S^*\left(1/\sqrt{2}\right)$

\[ = (f_b^a + \delta f_b^a) \left( B_a^h + \partial_a \xi^h \right), \]

that is,

\[ (f_i^h \partial_{\xi^h} f_i^h - f^a_i \partial_{\xi^a} f_i^h) - \xi^a u^a + u_b \xi^b - k_i \xi^i \partial_{\xi^j} f_i^j + k_j \xi^j \partial_{\xi^i} f_i^j) \varepsilon = (\delta f_b^a) B_a^h. \]

Thus substituting (2.4), (1.20) and (2.11) into this equation, we have

\[ \left[ (f_i^h \partial_{\xi^h} f_i^h - f^a_i \partial_{\xi^a} f_i^h) - \xi^a u^a + u_b \xi^b - k_i \xi^i \partial_{\xi^j} f_i^j + k_j \xi^j \partial_{\xi^i} f_i^j \right] \varepsilon = \left( \delta f_b^a \right) B_a^h. \]

or using (3.2) and (3.4),

\[ \delta f_b^a = \left[ (f_i^h \partial_{\xi^h} f_i^h - f^a_i \partial_{\xi^a} f_i^h) - \xi^a u^a + u_b \xi^b - k_i \xi^i \partial_{\xi^j} f_i^j + k_j \xi^j \partial_{\xi^i} f_i^j \right] \varepsilon, \]

or, using (3.3),

\[ \left( \xi^a \partial_{\xi^a} f_i^a \right) f_j^a - f^a_i \partial_{\xi^a} f_i^a - \xi^a u^a + u_b \xi^b - k_i \xi^i \partial_{\xi^j} f_i^j + k_j \xi^j \partial_{\xi^i} f_i^j = 0. \]

Thus we have

**Theorem 3.1.** In order for an infinitesimal variation (2.1) to carry an $f$-invariant submanifold $M^m$ of $S^*\left(1/\sqrt{2}\right) \times S^*\left(1/\sqrt{2}\right)$ into an $f$-invariant one, it is necessary and sufficient that (3.8) holds, the variation of $f_b^a$ being given by (3.7).

An infinitesimal variation given by (2.1) is called an $f$-invariance preserving variation if it carries an $f$-invariant submanifold into an $f$-invariant submanifold. If an $f$-invariance preserving variation preserves $f_b^a$ then we say that it is $f$-preserving.

From (1.7), (3.1) and (1.20), we find

\[ \left( \delta f_b^a \right)^2 = -\delta b^a + u_b u^a + v_b v^a, \]

\[ \left( f_b^a \right)^2 g_{ab} = g_{cb} - u_c u_b - v_c v_b, \]

\[ f_b^a u^b = -\lambda u^a, \quad f_b^a v^b = \lambda u^a, \]

\[ f_b^a v^b = \lambda u^a, \]
(3.12) \[ u_a u^a = 1 - \lambda^2 - u_x u^x, \quad v_a v^a = 1 - \lambda^2 - v_x v^x, \]
(3.13) \[ u_a v^a = - u_x v^x, \]
(3.14) \[ u_x u_b = - v_x v_b, \]
(3.15) \[ f_x f^z = - \delta_x^z + u_x u^z + v_x v^z, \]
(3.16) \[ u_x f_{x y} = - \lambda v_y, \quad v_x f_{x y} = \lambda u_y. \]

Equations (3.9)~(3.13) show that a necessary and sufficient condition \( f, g, u, v, \lambda \) to define \( (f, g, u, v, \lambda) \)-structure is that

\[ (3.17) \quad u_x = 0, \quad v_x = 0, \]

that is, the vector \( u^h \) and \( v^h \) are always tangent to the submanifold \( M^m \).

Transvecting (3.4) with \( f^b \), we get

\[ (3.18) \quad f_x f_{e f} h_{b e} = - h_{d e} x + (u_d u^e + v_d v^e) h_{e c} x + \lambda u_d k_c x + v_x f^d h_{b e}. \]

Now we assume that \( M^m \) is \( k \)-invariant submanifold of \( S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2}) \).

We then have

\[ (3.19) \quad k^b B_b^i = k_b^a B_a^h, \quad k^b C^i_j = k^x_j C^x_h. \]

Differentiating the first equation of (3.19) covariantly along \( M^m \) and using (1.9) and (3.19), we find

\[ (3.20) \quad \nabla_c k_b^a = 0, \quad h_{c b} x = k_b^e h_{c a} x. \]

Differentiating the second equation of (3.19) covariantly along \( M^m \) and using (1.9) and (3.19), we have

\[ (3.21) \quad \nabla_c k_c^z = 0, \quad k^h_{c e} k^z_{e x} = k^x_x k^h_{c e}, \]

the second equation is equivalent to the second equation of (3.20). From (1.21), we have

\[ (3.22) \quad k_b^a u^b = - v^a, \quad k_b^a v^b = - u^a, \quad k_b^a k^c_a = \delta_b^c. \]

We now consider an infinitesimal variation (2.1) and assume that it carries the \( k \)-invariant submanifold \( M^m \) of \( S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2}) \) into a \( k \)-invariant submanifold. Then we have

\[ k^h (x + \xi^e) B_b^i = (k_b^a + \delta k_b^a) B_a^h, \]

that is,

\[ (k^h + \xi^i \delta_j k_j^h) (B_b^i + \partial_b \xi^i \epsilon) = (k^a + \delta k_b^a) (B_a^h + \partial_a \xi^h \epsilon), \]

from which, using (1.9), we obtain
Infinitesimal variations of submanifolds of $S^*(1/\sqrt{2}) \times S^*(1/\sqrt{2})$

\[ [k_i^h + \xi^i ( - \Gamma_{ji}^h k_j^h + \Gamma_{ji}^h k_j^h )] \varepsilon = (k_b^a + \delta k_b^a) (B_a^h + \partial_{ab} \xi^h) \varepsilon, \]

that is,

\[ [k_i^h (V_b^a) - k_b^a V_{ab}^h] \varepsilon = (\delta k_b^a) B_a^h. \]

Thus substituting (2.4) and (2.11) into this equation, we have

\[ \begin{align*}
&[ (V_b^a - h_{bc}^e \xi^e) k_e^h ] B_a^h + (V_b^a + h_{bc}^e \xi^e) k_e^h C_x^h \\
&- \left[ (V_b^a - h_{bc}^e \xi^e) k_b^c B_a^h \right] (V_c^e + h_{ce}^e \xi^e) k_b^c C_x^h \varepsilon \\
&= (\delta k_b^a) B_a^h,
\end{align*} \]

from which,

\[ \delta k_b^a = \left[ \begin{align*}
&\left[ (V_b^a - h_{bc}^e \xi^e) k_e^h \right] k_e^a - (V_b^a - h_{bc}^e \xi^e) k_b^c \varepsilon \\
&\left[ \begin{align*}
&V_b^a - h_{bc}^e \xi^e \end{align*} \right] - k_b^c \left[ \begin{align*}
&V_b^c \xi^a \\
&h_{cb}^e \xi^e \\
&k_b^c \xi^e \end{align*} \right] = 0,
\end{align*} \]

or, using (3.21),

\[ \delta k_b^a = \left[ \begin{align*}
&\left[ (V_b^a - h_{bc}^e \xi^e) k_e^h \right] k_e^a - (V_b^a - h_{bc}^e \xi^e) k_b^c \varepsilon \\
&\left[ \begin{align*}
&V_b^a - h_{bc}^e \xi^e \end{align*} \right] - k_b^c \left[ \begin{align*}
&V_b^c \xi^a \\
&h_{cb}^e \xi^e \\
&k_b^c \xi^e \end{align*} \right] = 0,
\end{align*} \]

or, using the second equation of (3.20),

\[ (V_b^a - h_{bc}^e \xi^e) k_e^h - k_b^c (V_b^c \xi^a + h_{cb}^e \xi^e) = 0. \]

An infinitesimal variation given by (2.1) is called a $k$–invariance preserving variation if it carries a $k$–invariant submanifold into a $k$–invariant submanifold. Thus we have

**Theorem 3.2.** If an infinitesimal variation is $k$–invariance preserving, it is necessary and sufficient that the variation vector $\xi^b$ satisfies (3.25), the variation of $k_b^a$ being given by (3.23).

From (3.24) and (2.5), we have

**Theorem 3.3.** If the variation (2.1) of $k$–invariant submanifold is parallel, then it is $k$–invariance preserving.

4. Isometric variation and $f$–preserving variation of an $f$–invariant and $k$–invariant submanifold with induced $(f, g, \nu, \lambda)$–structure.

Applying the operator $\delta$ to $g_{cb} = g_{ji} B_c^i B_b^j$ and using (2.2), (2.4) and $\delta g_{ji} = 0$, we find [2]

\[ \delta g_{cb} = (V_b^a + V_b^a - 2 h_{cb} \xi^c) \varepsilon, \]
from which,

\[ \delta g^{ba} = - (V^b_x \xi_a + V^a_x \xi_b - 2 h^{ba} \xi_e) \epsilon. \]  

A variation of a submanifold for which \( \delta g_{cb} = 0 \) is said to be isometric. By a straightforward computation, we obtain

\[ \delta \Gamma_{cb}^a = \left[ (V_c \Gamma_b^c + K_{bc}^d \xi_d) B^a_{c} + h_{cb} (V^a \xi_c + h_{a} \xi_d) \right] \epsilon, \]

from which, using equations of Gauss and Codazzi of the submanifold \( M^m \), we have

\[ \delta \Gamma_{cb}^a = \left[ (V_c \Gamma_b^c + K_{bc}^d \xi_d) B^a_{c} + h_{cb} (V^a \xi_c + h_{a} \xi_d) \right] \epsilon. \]

A variation of a submanifold for which \( \delta \Gamma_{cb}^a = 0 \) is said to be affine.

Suppose that the submanifold is \( f \)-invariant and \( k \)-invariant, and has induced \((f, g, u, v, \lambda)\)-structure. Then, from (3.18), (2.9) and (2.16), we have

\[ f_{d} \partial_{c}^a h_{bc}^d = - h_{dc}^d, \quad u^b h_{cbx} = v^b h_{cbx} = 0. \]

If the variation of the submanifold is normal, we have from (3.7) and (4.1)

\[ \partial f_{b}^a = (f_{c} \partial_{c}^a h_{bc}^d + f_{c} a_{bc}^d \xi_c) \epsilon, \]

\[ \delta g_{cb} = - 2 h_{cbx} \xi_e \epsilon. \]

Thus we have from (3.9), (4.5), (4.6) and (4.7)

**Theorem 4.1.** Suppose that the variation of \( f \)-invariant and \( k \)-invariant submanifold with induced \((f, g, u, v, \lambda)\)-structure of \( S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2}) \) is normal. Then the variation is isometric if and only if it is \( f \)-preserving.

From the first equation of (4.5), we have

**Proposition 4.2.** An \( f \)-invariant and \( k \)-invariant submanifold with induced \((f, g, u, v, \lambda)\)-structure of \( S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2}) \) is minimal.

Furthermore we have from (3.9)

\[ \delta (f_{b}^a \xi_c) = (\partial u_b)^a \xi^c + u_b (\partial \xi^c) + (\partial v_b)^a \xi^c + v_b (\partial \xi^c). \]

If the variation preserves \( f_{b}^a \) and \( u^a \), we have from (4.9)

\[ (\partial u_b)^a \xi^c + (\partial v_b)^a \xi^c + v_b (\partial \xi^c) = 0. \]

Transvecting (4.10) with \( u_c \) and \( v_c \), we find respectively

\[ (1 - \lambda^2) \delta u_b + u_c v_b (\partial \xi^c) = 0, \]
Infinitesimal variations of submanifolds of $\mathbb{S}^{n(1/\sqrt{2})} \times \mathbb{S}^{n(1/\sqrt{2})}$

(4.12) $(1 - \lambda^2) \delta v_b + v_c v_b (\delta v^c) = 0$.

Transvecting (4.11) with $u^b$, $(\partial u_b) u^b = 0$. Then

$$\delta (u^b u_b) = -2\lambda (\delta \lambda) = 0,$$

that is, $\delta \lambda = 0$.

Applying the operator $\delta$ to (3.11), we can get $\delta v^a = 0$ from above. So from (4.11) and (4.12), $\delta u_b = 0$ and $\delta v_b = 0$. Thus we have

**Proposition 4.3.** If an infinitesimal $f$-preserving variation of the submanifold with induced $(f, g, u, v, \lambda)$-structure preserves $u^a$, then the variation preserves $u_a, v^a, v_a$ and $\lambda$.

Finally, when the submanifold with induced $(f, g, u, v, \lambda)$-structure is $f$-invariant and $k$-invariant, we get from the Ricci-identity, (1.17), (2.9), (2.16), (3.2) and (3.20)

(4.13) $K_{debc} f^e f^c = -g_{cb} f^d + g_{db} f^c + \delta_c f^d - \delta_b f^c$

$$+ k^c (f_{cb} k_c - k_{cb} f^c) + k^e (f_{de} k_{db} - k_{de} f^d).$$

Transvecting (4.13) with $f^d_a$ and using (3.9), (3.22) and (1.17), we get

(4.14) $K_{deca} f^c f^d = -K_{cb} + (u_a u^a + v_c v^c) K_{decb} + (m - 4 + 2\lambda^2) g_{cb}$

$$+ 2(u_a u_b + v_c v_b) + k_{cb}.$$ 

Transvecting (1.17) with $f^d_a$, we have

(4.15) $K_{decb} = 0$.

From (4.14) and (4.15), we find

(4.16) $K_{deca} f^c f^d = -K_{cb} + (u_a u^a + v_c v^c) K_{decb}$

$$+ (m - 4 + 2\lambda^2) g_{cb} + 2(u_a u_b + v_c v_b).$$

Differentiating $\lambda$ covariantly along $M^m$ and using the last equation of (2.8), we get

(4.17) $\Gamma^c \lambda = -2v_c$.

Using the Ricci-identity, (2.9), (2.16), (3.2), (3.20), (3.22) and (4.17), we obtain

(4.18) $K_{decb} (u^c u^d + v^c v^d) = 2(1 - \lambda^2) g_{cb} - 2(u_a u_b + v_c v_b)$.

Thus we can get the following useful identity from (4.16) and (4.18) for later use

(4.19) $K_{deca} f^c f^d = -K_{cb} + (m - 2) g_{cb}$. 

5. Some integral formulas.

In this section, on a compact $f$-invariant and $k$-invariant submanifold with induced $(f, g, u, v, \lambda)$-structure, we investigate some variation-preserving relations. Through this section, we assume that the submanifold $M^m$ of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ has induced $(f, g, u, v, \lambda)$-structure and is $f$-invariant and $k$-invariant.

First of all, we define $T_{b\alpha}$ by

$$T_{b\alpha} = (V_{b\xi} - h_{b\xi} \xi) f_{\alpha a} - f_{b\xi}(V_{\xi} a - h_{\xi a} \xi)$$

$$- \xi_b u_a + u_{b\xi} \xi_k b v_a + v_{b\xi} k e a.$$

Then we find that a variation of the submanifold preserves $f_{b\alpha}$ if and only if $T_{c\beta} = 0$.

If we take account of (3.9), (3.11), (3.22), (2.9), (2.16) and (3.18), we have

$$T_{b\alpha} T_{b\alpha} = 2(V_{b\xi} a)(V_{b\xi} c) - 8h_{b\xi} \xi(V_{\xi} b) + 4(h_{b\xi} \xi(V_{\xi} b))
- (u^a u_a + v^a v_a)[(V_{b\xi} a)(V_{b\xi} c) + (V_{\xi} b)(V_{\xi} a)]
- 2\lambda \xi_a \xi_b[(V_{b\xi} a) - (V_{a\xi} b)] + 2u_{b\xi} f a(V_{b\xi} - V_{c\xi} b)
- 2f_{b\xi} f_{\xi a}(V_{b\xi} a)
+ 2\lambda u_{b\xi} k_{eb}(V_{b\xi} b) + 2v_{b\xi} k_{eb} f a(V_{b\xi} c) - V_{\xi} b)
+ 4(1 - \lambda^2) \xi_a \xi c - 4(u^a \xi a)^2 + 4(v^a \xi a)^2.$$

From (3.2), (3.22) and (4.15), we get

$$\nabla_b f_{b\xi} = -mu^a.$$

On the other hand, we have

$$\nabla_b W_b = (V_{b\xi} b)(V_{\xi} b) + (V_{b\xi} \xi)(V_{\xi} b)
+ 2\kappa f_{b\xi} f_{\xi a}(V_{b\xi} a) + u^a f_{b\xi} f_{\xi a}(V_{\xi} \xi)
- u^a \xi a f_{\xi e}(V_{\xi} \xi)
+ f_{b\xi} k_{eb}(V_{b\xi} e - V_{a\xi} a V_{\xi} e)
- f_{b\xi} f_{b\xi}(V_{b\xi} e)(V_{\xi} \xi)
- f_{b\xi} f_{b\xi} f_{b\xi} \xi a - f_{b\xi} f_{b\xi} f_{b\xi} \xi a,$$

because of (3.2) and (4.15), where we have put

$$W_b = (V_{b\xi} \xi - f_{b\xi} f_{b\xi} \xi)(V_{\xi} \xi),$$

from which, using the Ricci-identity and (4.19),
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\[ V_b W^b = (V_b V^b) \xi_c + (V^b \xi_c) (V_b \xi_c) \]
\[ + m u f a \xi_a (V_c \xi_c) + u f a \xi_a (V_c \xi_c) \]
\[ - f c e u a \xi_a (V_c \xi_c) \]
\[ + f b c k^c \xi_c \xi_a V a \xi_a - V a \xi_a (V_b \xi_c) \]
\[ + K_{c b} \xi_c \xi_b - (m - 2) \xi_c \xi_c - f b c f a c (V_c \xi_c) (V_b \xi_a). \]

Comparing (5.2) with (5.5), we have

\[ T_{b a} T_{b a} = 2 (V_b W) - 2 \xi_c (V_b V^c + K_{c b} \xi^c) \]
\[ + 8 h c^b \xi_c (V_c \xi_b) + 4 \left( h c b^c \xi^c \xi^b \right) \]
\[ - (w^e u_a + v^f v_a) \left( (V_b \xi_c) (V^b \xi^a) + (V^b \xi^a) (V_b \xi^c) \right) \]
\[ - 2 \lambda v a \xi_b \left( (V^b \xi^a - V a \xi^b) \right) + 2 \lambda u a \xi^c \xi^b \xi^c \]
\[ - 2 (m + 1) u f a \xi_a (V_c \xi_c) + 2 v a \xi^c \xi^b \xi^c \xi^c \]
\[ + 2 (m - 2) \xi_c \xi_c - 4 \left( (\xi^c u_a)^2 - (\xi^c v_a)^2 \right). \]

or equivalently

\[ T^b_{a b} T^b_{a b} = 2 V_b (W^b - 2 h c^b \xi^c \xi^c) \]
\[ - 2 \xi_c [V^b V^b \xi_c + K_{c b} \xi^b - 2 V^b \left( h c b \xi^c \xi^c \right) \]
\[ - 2 h c^b \xi^c (V^b \xi^c + V^c \xi^b - 2 h c b \xi^c \xi^c) \]
\[ - (w^e u_a + v^f v_a) \left( (V^b \xi^a) (V^b \xi^c) + (V^b \xi^c) (V^b \xi^a) \right) \]
\[ - 2 \lambda v a \xi^c \xi^b \xi^c \xi^c + 2 \lambda u a \xi^c \xi^c \xi^c \xi^b \]
\[ - 2 (m + 1) f a a \xi^a \xi^a (V^c \xi^c) + 2 v a \xi^c \xi^b \xi^b \xi^c \]
\[ + 2 f c (V^c \xi^c) u a \xi^a + 2 v a \xi^a f b c \xi^c (V^c \xi^c) \]
\[ + 2 (m - 2) \xi^a \xi^a - 4 \left( (\xi^b u_a)^2 - (\xi^b v_a)^2 \right). \]

Thus we assume that the submanifold $M^n$ is compact. Using

\[ V_a (u a \xi^a \xi^c f a c) = - \xi^c \xi^c + (m+1) (u a \xi^a)^2 + (v a \xi^a)^2 \]
\[ + u a \xi^a f a c (V^c \xi^c) - \lambda \xi^c \xi^a \xi^b \xi^c \]
\[ + u a \xi^a f a c (V^c \xi^c) \]

and

\[ V_a (v a \xi^a \xi^c f b c k^c) = \xi^c \xi^c - (u a \xi^a)^2 - (v a \xi^a)^2 \]
\[ + \lambda \xi^c \xi^a \xi^a \xi^c \xi^b \xi^c + v a \xi^a f b c \xi^c (V^c \xi^c) \]
and (5.7), we apply Green's theorem and obtain

\[\int_{M^m} [T^b a T_{ba} + 2 \xi_c (V^b V_{b c}^e + K_{eb} \xi^b - 2V^b (h_{eb} \xi^e))] + 2h_{eb} \xi^e (V_{eb}^e + V_{eb}^e - 2h_{eb} \xi^e) + (u^e u_a + v^e v_a) \{(V_{eb}^e) (V_{eb}^e) + (V_{eb}^e) (V_{eb}^e)\}

+ 2\lambda u_{a} \xi_{a} (V_{eb}^e - V_{eb}^e) + 2\lambda u_{b} \xi_{b} (V_{eb}^e - V_{eb}^e) + 2(m + 1) f^{ace} \xi_{a} \xi_{u} (V_{eb}^e) + 2f^{cde} \xi_{d} \xi_{b} (V_{eb}^e)

+ 2v_{e} \xi_{e} k_{ac} f_{ac} (V_{eb}^e + V_{eb}^e) + 4\lambda \xi_{e} \xi_{e} k_{ac} f_{ac}

- 2(m - 2\lambda) \xi_{a} \xi_{a} + 2(m + 2) [(u^e \xi_{b})^2 - (v^e \xi_{b})^2)] dV = 0,\]

\(dV\) being the volume element of \(M^m\).

From (4.1) and (4.2), the variation of \(dV\) is given by [2]

\[\delta dV = (V_{eb}^e - h_{eb} \xi^e) dV e.\]

For a compact orientable submanifold \(M^m\), we have the following integral formula:

\[\int_{M^m} [\xi^c (V^b V_{b c}^e + K_{eb} \xi^b)] + \frac{1}{2} (V_{eb}^e + V_{eb}^e) (V_{eb}^e + V_{eb}^e) - (V_{eb}^e)^2 dV = 0,\]

which is valid for any vector \(\xi^e\) in \(M^m\) [4], from which

\[\int_{M^m} [\xi^c \{(V^b V_{b c}^e + K_{eb} \xi^b) - 2V^b (h_{eb} \xi^e) + V^c (h^b \xi^e)\} + \frac{1}{2} (V_{eb}^e + V_{eb}^e - 2h_{eb} \xi^e) (V_{eb}^e + V_{eb}^e - 2h_{eb} \xi^e)\]

\[- (V_{eb}^e - h_{eb} \xi^e) (V_{eb}^e) + (h_{eb} \xi^e) (V_{eb}^e + V_{eb}^e - 2h_{eb} \xi^e)]dV = 0.\]

Thus we have an integral formula from proposition (4.2), (5.8) and (5.10)

\[\int_{M^m} [T^b a T_{ba} + (u^e u_a + v^e v_a) \{(V_{eb}^e) (V_{eb}^e) + (V_{eb}^e) (V_{eb}^e)\} + 2\lambda u_{a} \xi_{a} (V_{eb}^e - V_{eb}^e) + 2\lambda u_{b} \xi_{b} (V_{eb}^e - V_{eb}^e) + 2(m + 1) f^{ace} \xi_{a} \xi_{u} (V_{eb}^e) + 2f^{cde} \xi_{d} \xi_{b} (V_{eb}^e)\]
Infinitesimal variations of submanifolds of $S^r(1/\sqrt{2}) \times S^s(1/\sqrt{2})$

\[+2v_b \xi^e \kappa_a \phi_{ab} (\nabla^b \xi^e + \nabla^e \xi^b) + 4 \lambda \kappa_a \phi_{ebra} \]
\[-2 (m - 2 \lambda^2) \xi^e \xi_a + 2 (m + 2) \left( (u^b \xi_b)^2 - (v^b \xi_b)^2 \right)\]
\[-(\nabla^b \xi^e + \nabla^e \xi^b - 2 h_{cb} \xi^c) \left( (V^e \xi^b + \nabla^b \xi^e - 2 h_{bd} \xi^d) \right)\]
\[+2 (\nabla^e \xi^e)^2 \] \[dV = 0.\]

Now we assume that the variation of the submanifolds preserves $u^a$, $v^a$, $u_a$ and $v_a$. Then we have from (2.12), (2.17) and (4.1)

\[(5.12) \quad u_b \nabla^b \xi^a = f^b \xi^b \xi_a - \lambda_k^b k^b \xi_a, \quad v_b \nabla^b \xi^a = \xi^e k_e^b f^a + \xi^a,
\]
\[(5.13) \quad u^a \nabla^a \xi_a = \lambda_k^b k^b - f_{bc} \xi^b, \quad v^a \nabla^a \xi_a = -\lambda_k^c - \xi_d f_{bc},\]

from which, we get

\[(5.14) \quad u_b \nabla^b \xi_b = -u_b \nabla^b \xi_a, \quad v_b \nabla^b \xi_b = -v_b \nabla^b \xi_a.\]

Using (5.11)~(5.14), we get

\[(5.15) \quad \int_M \left[ T^{ba} T_{ba} + 2 (m + 2) \lambda_k^a \kappa_a \phi_{ba} \right. \]
\[+ 4 \xi^a \xi_a - 4 (m + 2) (v_b \xi_b)^2 + 2 (\nabla^e \xi^e)^2 \]
\[-(\nabla^e \xi_b + \nabla^b \xi_e - 2 h_{cb} \xi_c) \left( (V^e \xi^b + \nabla^b \xi^e - 2 h_{bd} \xi^d) \right) \]
\[dV = 0.\]

To obtain the variation of $\lambda$, that is, $\delta \lambda = \lambda(x + \xi) - \lambda(x)$, using the last equation of (2.8), we have

\[(5.16) \quad \delta \lambda = -2 \xi^a v_a \xi.\]

By hypothesis, proposition (4.3) and (5.16), we find

\[\delta \xi_a = 0.\]

From (5.13), we have $\lambda_k^a = -v_b \nabla^b \xi_b - \xi_d f_{bc} \phi_{ba}$. Using the above equation and (5.17), we have

\[(5.18) \quad \lambda_k^a (\xi^2 k^e f_a) = -\xi^d \xi_a + (\xi^a u_a)^2 + \xi^e k^e f_a v_b (V^b \xi_a),\]
\[(5.19) \quad \lambda_k^a (\xi^e k^e f_a) = -\xi^d \xi_a + (\xi^a u_a)^2 + \xi^e k^e f_a v_b (V^b \xi_e).\]

We get from (5.18) and (5.19)

\[(5.20) \quad \xi^e k^e f_a v_b (V^b \xi_a) = \xi^e k^e f_a v_b (V^b \xi_e).\]

Using $\nabla_b (v^b \xi_a k^e f_e) = m \lambda_k^a \xi^2 k^e f_e + 2 (u^e \xi_e)^2 - 2 (v^b \xi_b)^2$
\[+ v_b (\nabla^e \xi_a) \xi^e k^e f_e + v_b (\nabla^b \xi_a) \xi^b k^e f_e\]
and (5.18), we apply Green's theorem and obtain
From (5.15), (5.17) and (5.22)

\[ \int_{M_m} \left[ T^{ba} T_{ba} - \nabla^2 (\nabla \xi)^2 \right] - 2 \nabla (\nabla \xi)^2 - \frac{1}{2} \nabla (\nabla \xi)^2 = 0. \]

From this integral formula, proposition 4.2 and (5.9), we have

**Proposition 5.1.** Suppose that a variation of the \( f \)-invariant and \( k \)-invariant compact submanifold with induced \( (f, g, u, v, \lambda) \)-structure of \( S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2}) \) preserves \( u_a, u_a, v_a \) and \( v_a \). Then in order for the variation to be isometric it is necessary and sufficient that the variation preserves volume and \( f_b \xi \).

Furthermore, if the variation of the submanifold is affine, we have from (4.4)

\[ \nabla (\nabla \xi)^2 = 0, \]

from which, using Proposition 4.2 \( \nabla (\nabla \xi)^2 = 0 \), that is, \( \xi_a = \text{constant} \).

Thus assuming the submanifold to be compact, we have \( \xi_a = 0 \). From this fact and proposition 5.1, we obtain

**Theorem 5.2.** Assume that the variation of a compact \( f \)-invariant and \( k \)-invariant submanifold with induced \( (f, g, u, v, \lambda) \)-structure of \( S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2}) \) preserves \( u_a, u_a, v_a \) and \( v_a \). Then the variation is isometric if and only if it is affine and preserves \( f_b \xi \).

Moreover, we have from proposition 4.3

**Corollary 5.3.** Suppose that the variation of a compact \( f \)-invariant and \( k \)-invariant submanifold with induced \( (f, g, u, v, \lambda) \)-structure of \( S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2}) \) preserves \( u \) and \( f_b \xi \). Then the variation is affine if and only if it is isometric.

6. Infinitesimal variations of \( f \)-antiinvariant submanifold and \( k \)-antiinvariant one.

We assume that \( M^m \) is an \( f \)-antiinvariant submanifold of \( S^n(1/\sqrt{2}) \times S^n \)
Infinitesimal variations of submanifolds of $S^t(1/\sqrt{2}) \times S^t(1/\sqrt{2})$.

We then have

\begin{equation}
(6.1) \quad f_i^h B_b^i = -f_b^i C_x^h, \quad f_i^h C_y^i = f_y^a B_a^h + f_y^x C_x^h.
\end{equation}

Differentiating the first equation of (6.1) covariantly along $M^m$, using (1.8) and (1.13), we find

\begin{align*}
&\left[ -g_{cb} u^a + u_b \partial_c^a - k_{cb} v^a + v_b k_c^a + h_{cb} f_x a \right] B_a^h \\
&\quad + \left[ -g_{cb} x - k_{cb} v^x + v_b k_c^x + h_{cb} f_y x \right] C_x^h \\
&\quad = f_b^x h_c^a x B_a^h - (\nabla_c f_b^x) C_x^h,
\end{align*}

and consequently, comparing the tangential and normal parts

\begin{equation}
(6.2) \quad h_c^a x f_x^a - h_c^a x f_y^x - g_{cb} u^a + \delta_c^a u_b - k_{cb} v^a + v_b k_c^a = 0
\end{equation}

and

\begin{equation}
(6.3) \quad \nabla_c f_b^x = g_{cb} u^x - h_b^x f_y^x + k_{cb} v^x - v_b k_c^x.
\end{equation}

From (6.2), taking the skew symmetric part, we have

\begin{equation}
(6.4) \quad h_c^a x f_y^x - h_b^a x f_x^x = \delta_c^a u_b - \delta_b^a u_c + v_b k_c^a - v_c k_b^a.
\end{equation}

Differentiating the second equation of (6.1) covariantly along $M^m$, using (1.8) and (1.13), we find

\begin{align*}
&\left( u_c \partial_e^a - k_{c,a} v^a + v_y k_c^a \right) B_a^h + \left( -k_{c,y} v^x + v_y k_c^x + h_{c,y} f_x x \right) C_x^h \\
&\quad = \left( -h_c^a x f_y^x + \nabla_c f_y^a \right) B_a^h + \left( f_y^a h_c^a x + \nabla_c f_y x \right) C_x^h.
\end{align*}

Thus, comparing the tangential and normal parts, we have

\begin{equation}
(6.5) \quad \nabla_c f_y^a = \delta_c^a u_y + h_e^a x f_y^x - k_{c,y} v^a + v_y k_c^a,
\end{equation}

which is equivalent to (6.3) and

\begin{equation}
(6.6) \quad \nabla_c f_y^x = -h_c^a x f_y^a - h_c^x f_y^x - k_{c,y} v^x + v_y k_c^x.
\end{equation}

We now consider an infinitesimal variation (2.1) and assume that it carries the $f$-antiinvariant submanifold $M^m$ into a $f$-antiinvariant submanifold. Then we have

\begin{equation}
(6.7) \quad f_i^h (x + \xi) \overline{B}_b^i = -(f_b^i + \delta f_b^i) \overline{C}_x^h,
\end{equation}

that is, using (2.6) and (2.7),

\begin{equation}
(6.8) \quad (f_i^h + \xi \partial_i f_i^h \xi) (B_b^i + \partial_b \xi^i \xi) \\
\quad = -(f_b^i + \delta f_b^i) [C_x^h - \Gamma_j^i h \xi^j C_x^i \xi + (\eta_x B_a^h + \eta_x X C_j^h) \xi],
\end{equation}

from which, using (1.8), we obtain
that is,

\[
\left[ f_i^h + \xi^j (-\Gamma^i_{ji} f_i^h + \Gamma^i_{ji} f_i^h - g_{ji} u^h + \delta^{ji} u_i - k_{ji} v^h + k_{ji} v_i) (B_i^h + \partial_i \xi^i e)\right] = - (f_y^a + \partial f_y^a) \left[ C_i^h - \Gamma^i_{ji} f_i^h C^j_e + (\eta_y^a B_i^h + \eta_y^a C^j_x) e \right],
\]

Thus substituting (2. 4) and (6. 1) into the above equation, we find

\[
\left[ f_i^h (V_b^e) - \xi_i^h u^h + u_y^h \right] e = - f_y^a (\eta_y^a B_a^h + \eta_y^a C_x^h) e - (\partial f_y^a) C_y^h + \left[ (\xi^i_k a^e + \xi^i_k b^e) v^a - v_b (\xi^i k_a^e + \xi^i k_x^a) \right] B_a^h e + \left[ (\xi^i_k a^e + \xi^i_k b^e) v^x - v_b (\xi^i k_a^x + \xi^i k_x^a) \right] C_x^h e,
\]

from which, comparing tangential and normal parts and using (2. 8),

\[
(6. 7) \quad (V_b^e) f_y^a = - f_y^a (V_a^e) + (\xi^i_b k^e + \xi^i_b b^e) v^a - v_b (\xi^i k_a^e + \xi^i k_x^a),
\]

which is, according to (6. 2), equivalent to

\[
(6. 8) \quad (V_b^e) f_y^a = f_y^a (V_a^e) + v^i (\xi^i k_b^e + \xi^i k_y^e) v^a - v_b (\xi^i k_a^e + \xi^i k_x^a),
\]

and

\[
(6. 9) \quad \partial f_y^a = \left[ (V_b^e - h_b^a \xi^a f_y^a) f_y^a - (V_b^e + h_b^a \xi^a) f_y^a + \xi^i k^a - u_y^a \partial f_y^a + (\xi^i k_b^e + \xi^i k_y^e) v^a - v_b (\xi^i k_a^e + \xi^i k_x^a) \right] e.
\]

Thus we have

**Theorem 6.1.** In order for an infinitesimal variation (2. 1) to carry an \( f \)-antiinvariant submanifold \( M^m \) of \( S^a (1/ \sqrt{2}) \times S^a (1/ \sqrt{2}) \) into \( f \)-antiinvariant one, it is necessary and sufficient that (6. 8) holds, the variation of \( f_b^a \) being given by (6. 9).

Now we assume that \( M^m \) is a \( k \)-antiinvariant submanifold of \( S^a (1/ \sqrt{2}) \times S^a (1/ \sqrt{2}) \). We then have

\[
(6. 10) \quad k_i^h B^i_b = k_b^e C_x^h, \quad k_i^h C^i_y = k_y^a B_a^h + k_y^a C_x^h.
\]
Infinitesimal variations of submanifolds of $S^*(1/\sqrt{2}) \times S^*(1/\sqrt{2})$

Differentiating the first equation of (6.10) covariantly along $M^m$ and using (1.8), we get

\begin{align}
(6.11) & \quad h_{cb}^x k_x^a = -h_{cx}^a k_b^x, \\
(6.12) & \quad \Gamma^c_{kb} x^a = h_{cb}^y k_y^x.
\end{align}

From (6.11), we find

\begin{equation}
(6.13) \quad h_{cb}^x k_x^a = 0.
\end{equation}

Differentiating the second equation of (6.1) covariantly along $M^m$ and using (1.8), we have

\begin{equation}
(6.14) \quad \Gamma^c_{ky} x^a = h_{cx}^a k_y^x,
\end{equation}

which is equivalent to (6.12) and

\begin{equation}
(6.15) \quad \Gamma^a_{ky} x^a = -k_y^a h_{ca}^x - h_{cy}^a k_x^a.
\end{equation}

We now consider an infinitesimal variation (2.1) and assume that it carries a $k$-antiinvariant submanifold $M^m$ into a $k$-antiinvariant one. Then we have

\[ k_i^h (x + \xi^i) \bar{B}_b^i = (k_b^x + \delta k_b^x) \bar{C}_{x}^{h}, \]

that is, using (2.6) and (2.7),

\[ (k_i^h + \xi^i \partial_j k_i^h) (B_b^i + \partial_b \xi^i) \]

\[ = (k_b^x + \delta k_b^x) \left[ C_{x}^{h} - \Gamma_{j i}^{h} \xi^j C_{x}^{i} \epsilon + (\eta_{x}^{a} B_{a}^{h} + \eta_{x}^{v} C_{y}^{a} \epsilon) \right], \]

from which, using (1.8), we obtain

\[ \left[ k_i^h + \xi^i (-\Gamma_{j i}^{h} k_i^h + \Gamma_{j i}^{h} k_i^h) \right] (B_b^i + \partial_b \xi^i) \]

\[ = (k_b^x + \delta k_b^x) \left[ C_{x}^{h} - \Gamma_{j i}^{h} \xi^j C_{x}^{i} \epsilon + (\eta_{x}^{a} B_{a}^{h} + \eta_{x}^{v} C_{y}^{a} \epsilon) \right], \]

that is,

\[ k_i^h (\bar{C}_b^i \xi^i) \epsilon = k_b^x (\eta_{x}^{a} B_{a}^{h} + \eta_{x}^{v} C_{y}^{a} \epsilon) + (\delta k_b^x) C_{x}^{h}. \]

Thus substituting (2.4) and (6.10) into the above equation,

\[ (\delta k_b^x) C_{x}^{h} = \left[ -k_b^x \eta_{x}^{a} + (\bar{C}_b^i \xi^i) k_y^a + \xi^i h_{by} \eta_{x}^{a} \right] B_{a}^{h} \epsilon \]

\[ + \left[ (\bar{C}_b^i \xi^i - h_{by} \eta_{x}^{a}) k_y^a + (\bar{C}_b^i \xi^i + h_{by} \eta_{x}^{a}) k_y^a - k_b^x \eta_{y}^{a} \right] C_{x}^{h} \epsilon, \]

from which, comparing tangential and normal parts, using (2.8) and (6.13),

\begin{align}
(6.16) & \quad k_b^x (\bar{C}_b^i \xi^i) + (\bar{C}_b^i \xi^i) k_y^a = 0, \\
(6.17) & \quad \delta k_b^x = [-k_b^x \eta_{y}^{a} + (\bar{C}_b^i \xi^i - h_{by} \eta_{x}^{a}) k_y^a].
\end{align}
Thus we have

**THEOREM 6.2.** In order for an infinitesimal variation \((2.1)\) to carry a \(k\)-antiinvariant submanifold \(M^n\) of \(S^a(1/\sqrt{2}) \times S^a(1/\sqrt{2})\) into a \(k\)-antiinvariant one, it is necessary and sufficient that \((6.16)\) holds, the variation of \(f_{\xi}^x\) being given by \((6.17)\).

An infinitesimal variation given by \((2.1)\) is called a \(k\)-antiinvariant variation if it carries a \(k\)-antiinvariant submanifold into a \(k\)-antiinvariant one.

Now from \((2.5)\), \((6.13)\) and \((6.16)\), we have

**THEOREM 6.3.** A parallel variation of a \(k\)-antiinvariant submanifold of \(S^a(1/\sqrt{2}) \times S^a(1/\sqrt{2})\) is \(k\)-antiinvariant variation.

On the other hand, from \((1.9)\), \((6.10)\) and \((6.11)\) we find

\[(6.18) \quad k_b^x k_x^y = \delta_b^a, \quad k_b^x k_x^y = 0, \quad k_y^x k_x^x + k_x^x k_x^x = \delta_y^x.\]

Assume that the variation of \(k\)-antiinvariant submanifold \(M^n\) preserves \(k_b^x\), we get from \((6.17)\)

\[(6.19) \quad -k_b^x \eta_x^y + (\nabla_{b} \xi^c - h_b^x \xi^y) k_c^x + (\nabla_{y} \xi^c + h_{bc} \xi^c) k_y^x = 0.\]

Then \((6.18)\) and \((6.19)\) imply

\[(6.20) \quad \nabla_{b} \xi^c - h_{xy} \xi^y = k_b^x k_x^x \eta_y^x.\]

Thus, by \((4.1)\), \((6.20)\) and \(\eta_{yx} = -\eta_{yx}\), \(\delta g_{ab} = 0\). Therefore we have

**THEOREM 6.4.** If the variation of \(k\)-antiinvariant submanifold of \(S^a(1/\sqrt{2}) \times S^a(1/\sqrt{2})\) preserves \(k_b^x\), then the variation is isometric.

**References**


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