THE INVARIANT DISTANCE DEFINED BY POSITIVE PLURIHARMONIC FUNCTIONS

DONG PYO CHI, IL HAE LEE, SA GE LEE, SANG MOON KIM

In this paper we study the pseudodistance defined by positive pluriharmonic functions.

Let \( M \) be a complex manifold. We call a real valued function \( u \) on \( M \) harmonic (=pluriharmonic) if, it is given locally by the real part of a holomorphic function. That is, for each \( z_0 \in M \), there is a holomorphic function defined on an open neighborhood \( V \) of \( z_0 \) with \( \Re(f(z)) = u(z) \) for all \( z \) in \( V \).

Let \( D \) be the open unit disk in the complex plane with the Kobayashi (=Poincare') pseudodistance \( k_D \). The Caratheodory pseudodistance \( c_D \) of \( M \) is defined by

\[
c_D(p, q) = \sup \{ k_D(f(p), f(q)) : f \in F \},
\]

where \( F \) denotes the set of holomorphic mappings \( f : M \to D \). It is well known that the Kobayashi pseudodistance \( k_D \) is equal to \( c_D \). For the definition and other relevant results about pseudodistances refer to [2] or [3].

Let \( H = \mathbb{R} : z = \{ x + iy, x > 0 \} \) be the right half plane with the Kobayashi pseudodistance \( k_H \). Define

\[
c_H(p, q) = \sup \{ k_H(g(p), g(q)) : g \in G \},
\]

where \( G \) denotes the set of all positive harmonic functions \( g \) on \( M \).

**Proposition 1.** Let \( D \) be the unit disk in the complex plane. Then the Kobayashi pseudodistance \( k_D \) coincides with the \( p_D \).

**Proof.** Let \( f : D \to H \) be the biholomorphic map such that \( f(0) = 1 \) and \( f(x) \geq 1 \) for \( 0 \leq x < 1 \). Let \( u \) be the real part of \( f \). Then it follows from the invariance of \( k_M \) for holomorphic maps that

\[
p_D(p, q) \geq k_H(u(p), u(q)) = k_D(p, q),
\]

for any two points in \((-1, 1) \cap D\).

Let \( A \) be a holomorphic automorphism of \( D \) that maps two points \( z_1 \) and
$z_2$ of $D$ to two points $p$ and $q$ in the real axis. Then we have the following relations;

$$p_D(z_1, z_2) \geq k_H(u(A(z_1)), u(A(z_2))) = k_D(p, q) = k_D(z_1, z_2).$$

In the above, equalities follow from the invariance of the Kobayashi pseudodistance for biholomorphic maps. Let $\alpha : D \to (0, \infty)$ be a positive harmonic function. Then we can define a holomorphic function $F : D \to H$, with it's real part $\alpha$. We shall show that

$$k_H(\alpha(p), \alpha(q)) \leq k_H(F(p), F(q)). \quad (2)$$

Let $I(H)$ be the group of biholomorphic transformations of $H$. Then $I(H)$ contains holomorphic maps, (a) $g(z) = k z$ ($k > 0$), and (b) $h(z) = z + it$ ($t = \text{real}$). Set $F(p) = \alpha(p) + i\beta(p)$ and $F(q) = \alpha(q) + i\beta(q)$. By the invariance of the pseudodistance $k_H$ for the transformations (a) and (b), we have the following equalities:

$$k_H(F(p), F(q)) = k_H(\alpha(p), \alpha(q) + i(\beta(q) - \beta(p))) = k_H(\alpha(p)/\alpha(p), \alpha(q)/\alpha(p) + i(\beta(q) - \beta(p))/\alpha(p)).$$

Hence in the proof of (2), we may assume that $\alpha(p) = 1$ and $\alpha(p) < \alpha(q)$. We shall prove (1), under the assumption $\alpha(p) = 1$, $\beta(p) = 0$, and $\alpha(p) < \alpha(q)$.

Let $A : H \to D$ be the biholomorphic map satisfying $A(1) = 0$, $A(z) = 0$ for $1 < x < \infty$.

Then we have

$$k_D(A(\alpha(p), A(\alpha(q))) = k_H(\alpha(p), \alpha(q)),
\quad k_D(A(F(p)), A(F(q))) = k_H(F(p), F(q)),
$$

and $A(\alpha(p)) = A(F(p)) = 0$ (since $\alpha(p) = F(p) = 1$).

There is the unique geodesic $\gamma$ (for the metric $k_D$) through $A(\alpha(q))$ and $A(F(q))$. The $\gamma$ is a part of the circle with center $c$ ($c > 1$). Since $A(\alpha(q)) > 0$, it is clear that

$$A(\alpha(q)) \leq |A(F(q))|,$$

and

$$k_D(0, A(\alpha(q)) \leq k_D(0, A(f(q))).$$

From this we have

$$k_H(\alpha(p), \alpha(q)) \leq k_H(F(q), F(q)).$$

Finally we have
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\[ k_D(p, q) = k_H(F(p), F(q)) \geq k_H(\alpha(p), \alpha(q)), \]

and

\[ k_D(p, q) \geq p_D(p, q). \]

Let \( \rho \) be a non-negative real valued function on \( M \times M \). If \( \rho(x, y) = \rho(y, x) \), \( \rho(x, y) + \rho(y, z) \geq \rho(x, z) \) and \( \rho(x, x) = 0 \) for any three points \( \{x, y, z\} \) in \( S \), then \( \rho \) is called a pseudometric (\( = \) pseudodistance) on \( S \).

We call any system which assigns a pseudodistance on each complex manifold a Schwarz–Pick system if it satisfies the following conditions:

(a) The distance assigned to the unit open disk in the complex plane is the Poincaré metric.

(b) If \( \rho_1 \) and \( \rho_2 \) are pseudometrics assigned to the complex manifolds \( S_1 \) and \( S_2 \) respectively, then \( \rho_2(h(x), h(y)) \leq \rho_1(x, y) \) for all holomorphic mappings \( h : S_1 \to S_2 \) and for each pair of points \( x \) and \( y \) in \( M \).

It is well known that the Kobayashi pseudodistance is the largest and the Caratheodory pseudodistance is the smallest one which can be assigned to complex manifolds by a Schwarz–Pick system.

**Definition.** Let \( p_M \) be the real valued function on \( M \times M \) defined by (1). We call it \( p_M \) pseudodistance.

It is clear by proposition 1 that the \( p_M \) pseudodistance satisfies all of the conditions of the Schwarz–Pick system.

**Proposition 2.** The \( p_M \) pseudodistance \( p_M \) is given by a Schwarz–Pick system.

There is a Riemann surface which carries a positive nonconstant harmonic function but it has no bounded analytic function. Hence the \( p_M \) is different from \( c_M \). The \( p_M \) is not a metric on compact Riemann surface. But the \( k_M \) is known to be a metric on a Riemann surface which is covered by the unit disk. From this we know that \( p_M \) and \( k_M \) are different.

We call a complex manifold \( M \) is \( p_M \) complete if, for each point \( p \) of \( M \) and each positive number \( r \), the closed ball of radius \( r (\{ q : p_M(p, q) \leq r, q \in M \}) \) is a compact subset of \( M \). In [3], Kobayashi defines the Carathéodory completeness. He proved that if \( M \) is \( c_M \) complete then \( M \) is \( F_\rho \) convex. Where \( F_\rho \) denotes the set of bounded holomorphic function \( f \) on \( M \) such that \( f(p) = 0 \). Let \( G \) be the set of positive harmonic functions on \( M \) and let \( G_\rho = \{ h \in G; h(p) = 1 \} \). We know from (2) that

\[ p_M(p, q) = \sup \{ k_H(h(p), h(q)) : h \in G \} = \sup \{ c_H(h(p), h(q)) : h \in G \} \] (3)
We define $\hat{h}(z) = \max \{ h(z), 1/h(z) \}$ for $h \in G_p$. Let $K$ be a subset of $M$, we set $\hat{k} = \{ z \in M : \hat{h}(z) \leq \sup \hat{h}(k) \text{ for } h \in G_p \}$. Then $\hat{k}$ is a closed subset of $M$, containing $K$. If $\hat{k}$ is compact for every compact subset of $M$, then $M$ is said to be convex with respect to $G_p$.

**Proposition 3.** Let $M$ be a complex manifold. Let $G_p$ be the set of positive harmonic function $h$ on $M$ such that $h(p) = 1$ and $p \in M$. If $M$ is complete for $p_M$ then $M$ is convex with respect to $G_p$.

**Proof.** Let $r$ be a positive number. Let $B(r)$ be the ball of radius $r$ around $p \in M$, that is,

$$B(r) = \{ q \in M : p_M(p, q) \leq r \}.$$ 

Since $M$ is complete, $B(r)$ is compact. Let $p \in K$ be a compact subset of $M$. Then

$$\sup \{ p_M(p, q) : q \in K \} \leq \sup \{ k_M(p, q) : q \in K \} = \alpha < \infty.$$ 

From the above we know that $K$ is contained in $B(r)$ for sufficiently large $r$. It suffices to show that $\hat{B}(r)$ is compact. By definition of $\hat{B}(r)$ we have

$$\hat{B}(r) = \{ q \in M : \hat{h}(q) \leq \sup_{t \in B(r)} \hat{h}(t), \ h \in G_p \} = \{ q \in M : |\log h(q)| \leq \sup_{t \in B(r)} |\log h(t)|, \ h \in G_p \}.$$ 

From (3) we know that

$$r = \sup_{t \in B(r)} |\log h(t)| : h \in G_p$$

and

$$p_M(p, q) = \sup \{ |\log h(q)| : h \in G_p \}.$$ 

It follows from the above

$$\hat{B}(r) = \{ q \in M : p_M(p, q) \leq r \}.$$ 

**Hence we have** $\hat{B}(r) = B(r)$.

In the following we study the $PH$ pseudodistance on Riemann surfaces. Let $M$ be a complex manifold and $p \in M$ then $M$ is said to be Carathéodory complete if for every $r > 0$ the closed ball of radius $r$ about $p$ in this distance is compact. Let $D$ be the open unit disk in the complex plane. Let $M_1 = \{ z : z \in D, \ z \neq 0 \}$. Then $-\log |z|$ is a positive harmonic function defined on $M_1$ and clearly it is not defined on $D$. Hence we have the following
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\[ \lim_{n \to \infty} p_{M_n}(1,1) = \infty, \]

and

\[ \lim_{n \to \infty} p_D(1,1) = p_D(1,0). \]

Let \( M_n = D - \{z_1, z_2, \ldots, z_n\} \). Since the Caratheodory pseudodistance is defined by bounded holomorphic functions, the extension property of bounded holomorphic function implies that \( c_{M_n}(p, q) = c_D(p, q) \) for \( p, q \in M_n \). From this we know that \( M_n(n > 0) \) is not complete for the pseudodistance \( c_{M_n} \). On the other hand (7) implies that \( M \) is complete for the \( PH \) pseudodistance. The same reasoning applies to \( p_{M_n} \) and hence \( M_n \) is also complete for \( p_{M_n} \). We may state the following proposition.

**Proposition 4.** Let \( D \) be the unit disk in the complex plane. Let \( M_n = D - \{z_1, \ldots, z_n\} \). Then \( M_n(n > 0) \) is not complete for the \( PH \) pseudodistance.

We shall give another example which shows the difference of \( c_M \) and \( p_M \). Let \( F_p \) be the set of bounded holomorphic function \( f \) on \( M \) with \( f(p) = 0 \). Kobayashi show that if \( M \) is Caratheodory complete then \( M \) is \( F_p \) convex. That is, if \( K \subseteq M \) is compact then so is

\[ \hat{k} = \{q \in M : |f(q)| \leq \sup_{t \in K} |f(t)|, f \in F_p\}. \]

Let \( x_n \) be a sequence of positive numbers converging monotonically to 0. Let \( r_n \) be another sequence of positive numbers such that the closed disks of radius \( r_n \) about \( x_n \) are pairwise disjoint and such that \( \sum r_n(x_n - r_n)^{-1} < \frac{1}{2} \).

Let \( N \) be the Riemann sphere with 0 and the union of these closed discs removed. In [1], they show that \( N \) is \( F_0 \) convex but not complete for the Caratheodory pseudometric. Let \( D \) be the open unit disk in the complex plane. Then there is a holomorphic one to one map \( f \) which maps \( N \) into \( D \). Set \( L = f(N) \) and \( a \) be in the boundary of \( L \). Then \( -\log \left| \frac{1}{2}(z-a) \right| \) is positive and harmonic on \( L \). From this it follows that \( L(N) \) is complete for the \( PH \) pseudodistance and convex for \( G_{f(\infty)}(G_{\infty}) \).

**References**


Seoul National University
Seoul 151, Korea