1. Introduction

Let $Q = [0, p] \times [0, q]$, where $p$ and $q$ are some fixed positive real numbers. Let $C_2[Q]$, called Yeh-Wiener space, denote the function space $\{x(\cdot, \cdot) \mid x(s, 0) = x(0, t) = 0, x(s, t) \text{ is a real valued continuous function on } Q\}$ with the uniform norm $\|x\| = \max_{(s, t) \in Q} |x(s, t)|$. Let $\mathcal{B}$ be the algebra of all subsets of $C_2[Q]$ of the form

$$I = \{x \in C_2[Q] \mid (x(s_1, t_1), \ldots, x(s_m, t_n)) \in E\}$$

where $m$ and $n$ are any positive integers, $0 = s_0 < s_1 < \cdots < s_m = p$, $0 = t_0 < t_1 < \cdots < t_n = q$ and $E$ is an arbitrary Lebesgue measurable set in the $mn$-dimensional Euclidean space $\mathbb{R}^{mn}$. Let $(C_2[Q], \mathcal{Q}, m_y)$, called the Yeh-Wiener measure space, denote the complete probability space where $\mathcal{Q}$ is the $\sigma$-algebra of Caratheodory measurable subsets of $C_2[Q]$ with respect to the outer measure induced by the probability measure $m_y$ on the algebra $\mathcal{B}$ defined for $I \in \mathcal{B}$ by

$$m_y(I) = \prod_{j=1}^m \prod_{k=1}^n \left\{2\pi (s_j - s_{j-1})(t_k - t_{k-1})\right\}^{-\frac{1}{2}} \int_E \exp \left\{-\frac{1}{2} \sum_{j=1}^m \sum_{k=1}^n \frac{(u_{jk} - u_{j-1,k} - u_{j,k-1} + u_{j-1,k-1})^2}{(s_j - s_{j-1})(t_k - t_{k-1})}\right\} \, du_{11} \cdots du_{mn}$$

where $u_{0,k} = u_{0,0} = u_{m,k} = 0 (j = 1, 2, \ldots, m, k = 1, 2, \ldots, n)$. (see [8]).

For a real valued Yeh-Wiener measurable (i.e. $\mathcal{Q}$-measurable) functional $F$ on $C_2[Q]$, the Yeh-Wiener integral of $F$ (i.e. the integral of $F$ with respect to $m_y$) will be denoted by

$$E^y[F] = \int_{C_2[Q]} F(x) \, dm_y(x)$$

whenever the integral exists. We say that $F$ is Yeh-Wiener integrable if $E^y[|F|] < \infty$. The Yeh-Wiener measurability and Yeh-Wiener integrability of a complex valued functional on $C_2[Q]$ are defined in terms of its real
and imaginary parts.

In [10], Yeh introduced the notion of conditional Wiener integral which is meant the conditional expectation $E_{w}(Z|X)$ of a real or complex valued Wiener integrable functional $Z$ conditioned by a Wiener measurable functional $X$ on the Wiener space which is given as a function on the value space of $X$. The organization of this paper is as follows:

In Section 2 we investigate some properties of the Paley–Wiener–Zygmund (P. W. Z.) integral over $Q$ by means of the stochastic integral and then prove a version of the P. W. Z. Theorem for $C_{2}[Q]$ which is an extension of Theorem 29.7 of [7] and the Yeh’s result proved in [12; Theorem II].

In Section 3 we define, following Yeh [10], a conditional Yeh–Wiener integral and state three inversion formulae for conditional Yeh–Wiener integrals (Theorems 3.3–3.5) which are analogous to the corresponding formulae for conditional Wiener integrals. We then give several examples of evaluation of conditional Yeh–Wiener integrals.

2. Notes on the P. W. Z. integral over $Q$

Let $Y$ be a real valued function on $Q \times C_{2}[Q]$ defined by

$$ Y((s, t), x) = x(s, t) \text{ for } ((s, t), x) \in Q \times C_{2}[Q]. $$

Then $Y$ is a measurable stochastic process on the probability space $(C_{2}[Q], \mathcal{F}, \mu)$ and the domain $Q$ in which the space of sample functions $Y(\cdot, x)$, $x \in C_{2}[Q]$, coincides with the sample space $C_{2}[Q]$. This stochastic process will be referred to as the Yeh–Wiener process on the domain $Q$. It is shown from the definition of $(C_{2}[Q], \mathcal{F}, \mathcal{F})$ that $Y((s, t), \cdot) \sim N(0, st)$ (i.e. normally distributed with mean $0$ and variance $st$) and $E Y(Y((u, \cdot), \cdot, Y((u, v), \cdot)) = \min \{s, u\} \cdot \min \{t, v\}$ for every $(s, t), (u, v) \in Q$.

**Definition 2.1.** A real valued function $f(s, t)$ is said to be of bounded variation on $Q$ (for symbols, $f \in BV[Q]$) if the following are satisfied:

(i) $f(s, 0)$ and $f(0, t)$ are of bounded variation on $[0, p]$ and $[0, q]$, respectively

(ii) The total variation $V(f)$ of $f$ on $Q$ is finite, where $V(f)$ is the supremum of

$$ \sum_{j=1}^{m} \sum_{i=1}^{n} |f(s_{i}, t_{j}) - f(s_{i-1}, t_{j}) - f(s_{i}, t_{j-1}) + f(s_{i-1}, t_{j-1})| $$

for any partition $P: 0 = s_{0} < s_{1} < \cdots < s_{m} = p$, $0 = t_{0} < t_{1} < \cdots < t_{n} = q$.

It is known [4] that if $f \in BV[Q]$, then $f$ is continuous almost everywhere on $Q$, so that $f$ is Riemann integrable over $Q$. 


For each \( n = 1, 2, \ldots \), let \( P_n \) denote a partition of \( Q \) given by
\[
0 = s_{n,0} < s_{n,1} < \cdots < s_{n,p(n)} = t_{n,q(n)} = q,
\]
with \( \lim_{n \to \infty} ||P_n|| = 0 \), where ||\( P_n || = \max_{i,j} \{ (s_{n,i} - s_{n,i-1}), (t_{n,j} - t_{n,j-1}) \} \).

For each partition \( P_n \) of \( Q \), let \( \xi_n \) denote a set of points in \( Q \) given by
\[
\xi_n = \{(u_{n,i}, v_{n,j}) \in [s_{n,i-1}, s_{n,i}] \times [t_{n,j-1}, t_{n,j}] \mid i = 1, 2, \ldots, p(n), \ j = 1, 2, \ldots, q(n) \}.
\]

For \( f \in BV[Q] \), let \( S(f, P_n, \xi_n)(x) \) denote the Riemann–Stieltjes sum of \( f \) with respect to \( Y \) defined by
\[
S(f, P_n, \xi_n)(x) = \sum_{i=1}^{p(n)} \sum_{j=1}^{q(n)} f(u_{n,i}, v_{n,j}) \Delta_{i,j}^n Y(x), \ x \in C_2[Q]
\]
where \( \Delta_{i,j}^n Y(x) = Y((s_{n,i}, t_{n,j}), x) - Y((s_{n,i-1}, t_{n,j}), x) - Y((s_{n,i}, t_{n,j-1}), x) + Y((s_{n,i-1}, t_{n,j-1}), x) \).

**Remark 2.1.** We note that \( S(f, P_n, \xi_n)(x) \) is, indeed, equal to the Riemann–Stieltjes sum of \( f \in BV[Q] \) with respect to \( x \in C_2[Q] \) corresponding to \( P_n \) and \( \xi_n \), and that it is known [11] that the Riemann–Stieltjes integral \( \int_Q f dx \) of \( f \in BV[Q] \) exists for every \( x \in C_2[Q] \). Hence \( \lim_{n \to \infty} S(f, P_n, \xi_n)(x) \) exists and is equal to \( \int_Q f dx \) for every \( x \in C_2[Q] \).

**Definition 2.2.** For \( f \in BV[Q] \), we define the *stochastic integral* \( I(f) \) of \( f \) with respect to the Yeh–Wiener process \( Y \) is defined by
\[
I(f)(x) = \lim_{n \to \infty} S(f, P_n, \xi_n)(x), \ x \in C_2[Q].
\]

We state the following lemma which can be proved by simple computations.

**Lemma 2.1.** With the same notation as in the above, we have the following:

1. \( \Delta_{i,j}^n Y(\cdot) \sim N(0, (s_{n,i} - s_{n,i-1})(t_{n,j} - t_{n,j-1})) \) for all \( i, j \) and \( n \).
2. \( E^y(\Delta_{i,j}^n Y(\cdot) \cdot \Delta_{k,l}^n Y(\cdot)) = 0 \) for all \( (i, j) \neq (k, l) \).

Let \( L^2[C_2[Q]] \) (\( L^2[Q] \)) denote the real Hilbert space of all square Yeh–Wiener integrable functions over \( C_2[Q] \) (Lebesgue integrable functions over \( Q \), resp.). We shall write \( \langle \cdot, \cdot \rangle \) and \( ||\cdot|| \) for inner product and norm in both spaces \( L^2[C_2[Q]] \) and \( L^2[Q] \) since there will be no ambiguity from the context. Since \( BV[Q] \subset L^2[Q] \), for any \( f, g \in BV[Q] \) we shall also use the notation \( \langle f, g \rangle \) and \( ||f|| \) for the inner product of \( f \) and \( g \), and the norm of \( f \) in the sense of \( L^2[Q] \) space respectively.
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**Proposition 2.2.** For \( f, g \in BV[Q] \) and \( \alpha, \beta \in R^1 \), the stochastic integral \( I \) satisfies the following:

1. \( I(f) \sim N(0, \|f\|^2) \).
2. \( \|I(f)\|^2 = \|f\|^2 \).
3. The sequence \( \{S(f, P_n \xi_n)\} \) converges in \( L^2[C_2[Q]] \) to \( I(f) \in L^2[C_2[Q]] \).
4. \( \langle I(f), I(g) \rangle = \langle f, g \rangle \).
5. \( I(\alpha f + \beta g) = \alpha I(f) + \beta I(g) \).

**Proof.** (1) Since \( S(f, P_n \xi_n) \) is a linear combination of normally distributed random variables \( Y((s, t), \cdot) \) on \( C_2[Q] \), it is normally distributed and by Lemma 2.1, we have \( S(f, P_n \xi_n) \sim N(0, \sum_{i=1}^{q(n)} \sum_{j=1}^{p(n)} |f(u_{ni}, v_{nj})|^2(s_{ni} - s_{ni-1})(t_{nj} - t_{nj-1})) \). Thus the characteristic function of \( S(f, P_n \xi_n) \), \( \phi_n(t) \) is given by

\[
\phi_n(t) = \exp \left\{ -\frac{t^2}{2} \sum_{i=1}^{q(n)} \sum_{j=1}^{p(n)} |f(u_{ni}, v_{nj})|^2(s_{ni} - s_{ni-1})(t_{nj} - t_{nj-1}) \right\}
\]

for every \( t \in R^1 \). Hence we have, every \( t \in R^1 \),

\[
\lim_{n \to \infty} \phi_n(t) = \exp \left\{ -\frac{t^2}{2} \|f\|^2 \right\} \equiv \phi(t).
\]

Since \( S(f, P_n \xi_n)(x) \) converges to \( I(f)(x) \) for every \( x \in C_2[Q] \), it follows from Levy's continuity Theorem [2; p.332] that \( \phi(t) \) is the characteristic function of \( I(f) \), so that we have \( I(f) \sim N(0, \|f\|^2) \).

(2) Since \( E^y(I(f)) = 0 \), the variance of \( I(f) \) is given by \( E^y(I(f)^2) \), so that we have \( \|I(f)\|^2 = \|f\|^2 \).

(3) From the definition of Riemann integral of \( f \), we have

\[
\lim_{n \to \infty} \|S(f, P_n \xi_n)\|^2 = \lim_{n \to \infty} \left\{ \sum_{i=1}^{q(n)} \sum_{j=1}^{p(n)} |f(u_{ni}, v_{nj})|^2(s_{ni} - s_{ni-1})(t_{nj} - t_{nj-1}) \right\}
= \int_0^1 |f(s, t)|^2 ds dt = \|f\|^2.
\]

Since \( \lim_{n \to \infty} S(f, P_n \xi_n)(x) = I(f)(x) \) for every \( x \in C_2[Q] \), it follows from [6; p.118] that \( S(f, P_n \xi_n) \) converges in \( L^2[C_2[Q]] \) to \( I(f) \).

(4) By using the result of (3), we have

\[
\langle I(f), I(g) \rangle = \lim_{n \to \infty} \langle S(f, P_n \xi_n), S(g, P_n \xi_n) \rangle
= \lim_{n \to \infty} \left\{ \sum_{i=1}^{q(n)} \sum_{j=1}^{p(n)} f(u_{ni}, v_{nj}) g(u_{ni}, v_{nj})(s_{ni} - s_{ni-1})(t_{nj} - t_{nj-1}) \right\}
= \int_0^1 f(s, t) g(s, t) ds dt = \langle f, g \rangle.
\]

(5) The proof follows from the fact

\[
S(\alpha f + \beta g, P_n \xi_n) = \alpha S(f, P_n \xi_n) + \beta S(g, P_n \xi_n).
\]
REMARK 2.2. (a) Let \( \{e_k \}_{k=1}^{\infty} \subset BV[Q] \) be a complete orthonormal set (C.O.N. set) for \( L^2[Q] \). For \( f \in L^2[Q] \), let \( f_n(s,t) = \sum_{k=1}^{\infty} a_k e_k(s,t) \) where \( a_k = \langle f, e_k \rangle \). Then since \( \{f_n\} \) converges in \( L^2[Q] \) to \( f \), we note from (2) of Proposition 2.2 that there exists the element \( \hat{I}(f) \) in \( L^2[C_2[Q]] \) to which \( \{I(f_n)\} \) converges in \( L^2[C_2[Q]] \). We also note from (2) of Proposition 2.2 that the element \( \hat{I}(f) \) in \( L^2[C_2[Q]] \) is determined by \( f \) independently of the choice of the C.O.N. set \( \{e_k\} \) for \( L^2[Q] \).

(b) Let \( \{e_k\} \) and \( \{a_k\} \) be as in (a). Since \( I(a_i e_i) \sim N(0, a_i^2) \) and covariance of \( I(a_i e_i) \) and \( I(a_j e_j) \) is zero for all \( i, j \) with \( i \neq j \), \( \{I(a_i e_i)\} \) is a sequence of independent random variables on \( C_2[Q] \) with \( \sum_{i=1}^{\infty} E^y(I(a_i e_i)^2) = \sum_{i=1}^{\infty} a_i^2 = \|f\|^2 < \infty \). Hence it follows from [3; p.197] that \( \lim_{n \to \infty} I(f_n)(x) \) exists for \( m_y \)-a.e. \( x \) in \( C_2[Q] \).

(c) We note from (a) and (b) that for \( f \in L^2[Q] \), \( \hat{I}(f)(x) = \lim_{n \to \infty} I(f_n)(x) \) for \( m_y \)-a.e. \( x \) in \( C_2[Q] \) and that for \( f \in BV[Q] \), \( \hat{I}(f)(x) = I(f)(x) \) for \( m_y \)-a.e. \( x \) in \( C_2[Q] \).

DEFINITION 2.3. For \( f \in L^2[Q] \), we define the stochastic integral of \( f \) with respect to the Yeh–Wiener process \( Y \) to be the element \( I(f) \) in \( L^2[C_2[Q]] \) of Remark 2.2.

The P. W. Z. integral of \( f \) over \( Q \) with respect to \( x \in C_2[Q] \) is defined to be the real number \( \hat{I}(f)(x) \) and will be denoted by

\[
\hat{I}(f)(x) = \int_Q f \, dx, \quad \text{or} \quad \int_Q f(s,t) \, dx(s,t).
\]

PROPOSITION 2.3. For \( f, g \in L^2[Q] \) and \( \alpha, \beta \in R^1 \), the stochastic integral \( \hat{I} \) satisfies (1), (2), (4) and (5) of Proposition 2.2.

Proof. To prove (1) let \( \{f_n\} \) and \( \{a_n\} \) be as in Remark 2.2. Since \( I(f_n) \sim N(0, \|f_n\|^2) = N \left( 0, \sum_{k=1}^{\infty} a_k^2 \right) \), by using the same argument as in the proof of (a) in Proposition 2.2, we have \( \hat{I}(f) \sim N \left( 0, \sum_{k=1}^{\infty} a_k^2 \right) = N(0, \|f\|^2) \).

The proofs of (2), (4) and (5) are similar to the corresponding proofs of (2), (4) and (5) in Proposition 2.2.

The following theorem is a slight extension of the P. W. Z. Theorem for \( C_2[Q] \) in [12; p.1430]. It may be noted that our proof of this theorem is rather simpler than that of Yeh [12].
THEOREM 2.4. Let \( \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \) be an orthogonal set in \( L^2[Q] \), and let \( f(u_1, \ldots, u_n) \) be a real or complex valued Lebesgue measurable function on \( \mathbb{R}^n \). Then the functional \( F \) on \( C_2[Q] \) defined by

\[
F(x) = f(\int_Q \alpha_1 dx, \ldots, \int_Q \alpha_n dx), \quad x \in C_2[Q],
\]

is Yeh–Wiener measurable and

\[
E^\gamma(F) = \{(2\pi)^{n/2} \prod_{j=1}^{n} ||\alpha_j||^2 \}^{-\frac{1}{2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(u_1, \ldots, u_n) \cdot 
\exp \left\{ -\frac{1}{2} \sum_{j=1}^{n} \frac{u_j^2}{||\alpha_j||^2} \right\} du_1 \cdots du_n
\]

where \( \gamma \) means that the existence of one side implies that of the other with the equality.

Proof. Let \( \hat{I}_n(x) = (\hat{I}(\alpha_1)(x), \ldots, \hat{I}(\alpha_n)(x)), \quad x \in C_2[Q] \). Then \( \hat{I}_n \) is an \( n \)-dimensional random vector on \( C_2[Q] \). Since \( \hat{I}(\alpha_i) \sim N(0, ||\alpha_i||^2) \) and \( \langle \hat{I}(\alpha_i), \hat{I}(\alpha_j) \rangle = \langle \alpha_i, \alpha_j \rangle = 0 \) for all \( i, j \) with \( i \neq j \), it follows that \( \{\hat{I}(\alpha_i) | i=1,2,\ldots, n\} \) is a set of independent random variables, so that the probability distribution of \( \hat{I}_n \) on \( \mathbb{R}^n \), \( m_\gamma \circ \hat{I}_n^{-1} \) is equal to \( m_\gamma \circ \hat{I}_n^{-1}(\alpha_1) \times \cdots \times m_\gamma \circ \hat{I}_n^{-1}(\alpha_n) \). Hence we have, for every Borel set \( B \) in \( \mathbb{R}^n \),

\[
m_\gamma(\hat{I}_n^{-1}(B)) = \left\{(2\pi)^{n/2} \prod_{j=1}^{n} ||\alpha_j||^2 \right\}^{-\frac{1}{2}} \int_B \exp \left\{ -\frac{1}{2} \sum_{j=1}^{n} \frac{u_j^2}{||\alpha_j||^2} \right\} du_1 \cdots du_n.
\]

Now let us show that \( \hat{I}_n \) is a measurable transformation of \( (C_2[Q], \mathcal{F}, m_\gamma) \) into \( (\mathbb{R}^n, \mathcal{B}) \) where \( \mathcal{B} \) denotes the \( \sigma \)-algebra of all Lebesgue measurable subsets of \( \mathbb{R}^n \). To show this we need to show that \( \hat{I}_n^{-1}(E) \in \mathcal{F} \) for every \( E \in \mathcal{B} \). Now let \( E \in \mathcal{B} \) and \( E = B \cup N_1 \), where \( B \) is a Borel set in \( \mathbb{R}^n \) and \( N_1 \) is a subset of a null Borel set \( N \) in \( \mathbb{R}^n \). Then \( \hat{I}_n^{-1}(B) \) and \( \hat{I}_n^{-1}(N) \) are in \( \mathcal{F} \). Since \( m_\gamma \) is complete and \( m_\gamma(\hat{I}_n^{-1}(N_1)) = 0 \) by (2.3), we have \( \hat{I}_n^{-1}(E) \in \mathcal{F} \). Hence if \( f \) is a Lebesgue measurable function on \( \mathbb{R}^n \), then \( f = f \circ \hat{I}_n \) defined by (2.1) is a Yeh–Wiener measurable functional on \( C_2[Q] \).

The formula in (2.2) can be easily obtained by using the change of variables theorem \([3; \text{p.163}]\) and (2.3).

3. Conditional Yeh–Wiener Integral

Let \( X \) be a real valued Yeh–Wiener measurable functional on \( (C_2[Q], \mathcal{F}, m_\gamma) \). The probability distribution of \( X \) is by definition a probability
measure $P_\pi$ on $(R^1, \mathcal{B}(R^1))$ given by

$$P_\pi(B) = m_\pi(X^{-1}(B)) \text{ for } B \in \mathcal{B}(R^1),$$

where $\mathcal{B}(R^1)$ denotes the $\sigma$-algebra of the Borel sets in $R^1$.

**Definition 3.1.** Let $X$ and $Z$ be real valued Yeh–Wiener measurable functionals on $(C_2[Q], \mathcal{B}, m_\pi)$ with $E^\pi[Z] \leq \infty$. The conditional Yeh–Wiener integral of $Z$ given $X$, written $E^\pi(Z|X)$, is defined to be the equivalence class of $\mathcal{B}(R^1)$–measurable and $P_\pi$–integrable functions $f$ on $R^1$ modulo $P_\pi$–a.e. equality on $R^1$ such that for $B \in \mathcal{B}(R^1)$,

$$\int_{X^{-1}(B)} Z(x) dm_\pi(x) = \int_B f(w) dP_\pi(w). \quad (3.1)$$

**Remark 3.1** By Radon–Nikodym Theorem a function $f$ on $R^1$ satisfying (3.1) exists, and if $g$ is another such function on $R^1$, then $f(w) = g(w)$ for $P_\pi$–a.e. $w \in R^1$. We shall also use $E^\pi(Z|X)$ to mean a particular version of the equivalence class. Thus the equation (3.1) can be written as

$$\int_{X^{-1}(B)} Z(x) dm_\pi(x) = \int_B E^\pi(Z|X)(w) dP_\pi(w). \quad (3.2)$$

The following results (Lemma 3.1, Corollary 3.2, Theorems 3.3–3.5) are simple modifications of the corresponding results for the Wiener space in [10]. The proofs in our setting remain the same.

**Lemma 3.1** Let $X$ and $Z$ be real valued Yeh–Wiener measurable functionals on $C_2[Q]$ with $E^\pi[Z] \leq \infty$. Then for any $\mathcal{B}(R^1)$–measurable function $g$ on $R^1$, we have

$$E^\pi[(g \cdot X)Z] = \int_{R^1} g(w) E^\pi(Z|X)(w) dP_\pi(w).$$

**Corollary 3.2.** Let $X$ and $Z$ be as in Lemma 3.1. Assume that $P_\pi$ is absolutely continuous with respect to Lebesgue measure $m$ on $(R^1, \mathcal{B}(R^1))$. For every $w \in R^1$ and $a > 0$, let $J^w_a$ be the function on $R^1$ defined by

$$J^w_a(\xi) = \begin{cases} \frac{1}{2a}, & \xi \in [w-a, w+a] \\ 0, & \xi \notin [w-a, w+a]. \end{cases}$$

Then there exists a version of $E^\pi(Z|X) \frac{dP_\pi}{dm}$ such that for $w \in R^1$,

$$E^\pi(Z|X)(w) \frac{dP_\pi}{dm}(w) = \lim_{a \to 0} E^\pi[(J^w_a \cdot X)Z] \quad (3.3)$$
where \( \frac{dP_x}{dm} \) denotes the Radon–Nikodym derivative of \( P_x \) with respect to \( m \).

**Theorem 3.3.** Let \( X \) and \( Z \) be real valued Yeh–Wiener measurable functionals on \( C_2[Q] \) with \( E^v[|Z|] < \infty \). Assume that \( P_x \) is absolutely continuous with respect to \( m \) on \( (R^1, \mathcal{B}(R^1)) \). Then there exists a version of \( E^v(Z|X) \frac{dP_x}{dm} \) such that for \( w \in R^1 \),

\[
E^v(Z|X)(w) \frac{dP_x}{dm}(w) = \lim_{h \to 0} \frac{1}{2\pi} \int_{(-h, h)} \left( 1 - \frac{|u|}{h} \right) e^{-iuw} E^v[e^{iuX}Z] dm(u).
\]

**Theorem 3.4.** Let \( X \) and \( Z \) be as in Theorem 3.3. For \( a, b \in R^1 \) with \( a < b \), let \( J_{a,b} \) be the function on \( R^1 \) defined by

\[
J_{a,b}(w) = \begin{cases} 
1, & w \in (a, b) \\
0, & w \in [a, b] \\
\frac{1}{2}, & w=a, w=b.
\end{cases}
\]

Then we have

\[
\int_{R^1} J_{a,b}(w) E^v(Z|X)(w) \frac{dP_x}{dm}(w) dm(w) = \lim_{h \to 0} \int_{(-h, h)} \frac{e^{-iuw} - e^{ibu}}{iu} E^v[e^{iuX}Z] dm(u).
\]

**Theorem 3.5.** Let \( X \) and \( Z \) be as in Theorem 3.3. Assume that \( E^v(e^{iuX}Z) \) is a Lebesgue integrable function of \( u \) on \( (R^1, \mathcal{B}(R^1)) \). Then there exists a version of \( E^v(Z|X) \frac{dP_x}{dm} \) such that for \( w \in R^1 \),

\[
E^v(Z|X)(w) \frac{dP_x}{dm}(w) = \frac{1}{2\pi} \int_{R^1} e^{-iuw} E^v[e^{iuX}Z] dm(u).
\]

As examples of evaluation of conditional Yeh–Wiener integrals we have the following:

**Example 1.** Let \( X \) and \( Z \) be real valued Yeh–Wiener measurable functionals on \( C_2[Q] \) defined by, respectively,

\[
X(x) = x(p, q) \quad \text{and} \quad Z(x) = \frac{1}{pq} \int_0^q \int_0^p x(u, v) dudv \quad \text{for} \ x \in C_2[Q].
\]

Then \( Z \) is Yeh–Wiener integrable since it can be shown that \( E^v[|Z|] = \frac{4\sqrt{2}}{6} \sqrt{\frac{pq}{\pi}} < \infty \) by applying the Fubini theorem. Hence \( E^v(Z|X) \) exists.
Now let us find $E^y(Z|X)(w)$, $w \in \mathbb{R}^1$. We first note that from the fact that $Y((p, q), \cdot) \sim \mathcal{N}(0, pq)$, we have

\begin{equation}
(3.4) \quad \frac{dP_y}{dm}(w) = \frac{1}{\sqrt{2\pi pq}} \exp\left(-\frac{w^2}{2pq}\right) \quad \text{for } w \in \mathbb{R}^1.
\end{equation}

By Corollary 3.2, there exists a version of $E^y(Z|X) \frac{dP_y}{dm}$ such that

\begin{equation}
(3.5) \quad E^y(Z|X)(w) \frac{dP_y}{dm}(w) = \lim_{a \to 0} E^y[(J^a_w X)^2] Z \quad \text{for } w \in \mathbb{R}^1.
\end{equation}

But with our $X$ and $Z$, we have

\begin{equation}
(3.6) \quad E^y[(J^a_w X)^2] Z = E^y[J^a_w x(p, q)] \frac{1}{pq} \int_0^p x(u, v) du dv.
\end{equation}

We apply the Fubini Theorem to the right-hand side of (3.6) to obtain

\begin{equation}
(3.7) \quad E^y[(J^a_w X)^2] Z = \frac{1}{pq} \int_0^p E^y[J^a_w x(p, q)] x(u, v) du dv.
\end{equation}

Evaluating the integrand of the integral of the right-hand side of (3.7) with the long and tedious computations (see [1]), we obtain

\begin{equation}
(3.8) \quad E^y[J^a_w x(p, q)] x(u, v)] = \frac{1}{2a} \frac{1}{(2\pi)^2 uv} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-\frac{u_{11}^2}{2uv} - \frac{(u_{12} - u_{11})^2}{2u(q-v)} - \frac{(u_{21} - u_{11})^2}{2v(p-u)}\right) du_{21} du_{12} du_{11} du_{22}
= \frac{1}{2a pq} \frac{uv}{2\pi pq} \int_{-\infty}^{\infty} \exp\left(-\frac{u_{22}^2}{2pq}\right) du_{22}.
\end{equation}

Substituting (3.8) for the right-hand side of (3.7), we obtain

\begin{equation}
(3.9) \quad E^y[(J^a_w X)^2] Z = \frac{1}{4} \frac{1}{\sqrt{2\pi pq}} \frac{1}{2a} \int_{-\infty}^{\infty} \exp\left(-\frac{u_{22}^2}{2pq}\right) du_{22}.
\end{equation}

Combining (3.5) and (3.9), we have

\begin{equation}
E^y(Z|X)(w) \frac{dP_y}{dm}(w) = \lim_{a \to 0} \left\{ \frac{1}{4} \frac{1}{\sqrt{2\pi pq}} \frac{1}{2a} \int_{-\infty}^{\infty} \exp\left(-\frac{u_{22}^2}{2pq}\right) du_{22}\right\}
= \frac{w}{4} \frac{1}{\sqrt{2\pi pq}} \exp\left(-\frac{w^2}{2pq}\right), \quad w \in \mathbb{R}^1.
\end{equation}

Hence it follows from (3.4) that we obtain

\begin{equation}
E^y(Z|X)(w) = \frac{w}{4} \quad \text{for } w \in \mathbb{R}^1.
\end{equation}
EXAMPLE 2. Let $X$ and $Z$ be real valued Yeh–Wiener measurable functionals on $C_2[Q]$ defined by, respectively,

$$X(x) = x(p, q) \quad \text{and} \quad Z(x) = \int_0^q \int_0^p [x(u, v)]^2 \, du \, dv \quad \text{for} \quad x \in C_2[Q].$$

We first note that $Z$ is Yeh–Wiener integrable and $E[Z] = \frac{(pq)^2}{4}$. Thus $E^x[Z \mid X](w)$ exists. Let us find $E^x[Z \mid X](w)$ in the way that we proceeded in Example 1. To apply Corollary 3.2, observe that

$$E^x[J_a \cdot X] Z = E^x[J_a \cdot X](x(p, q)) \int_0^q \int_0^p [x(u, v)]^2 \, du \, dv = \int_0^q \int_0^p E^x[J_a \cdot X](x(p, q)) [x(u, v)]^2 \, du \, dv.$$

Evaluating the integrand of the integral of the right–hand side of (3.10) with the long and tedious computations (See [1]), we obtain

$$E^x[J_a \cdot X](x(p, q)) [x(u, v)]^2$$

$$= \frac{1}{2a} \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u_{11}^2 \exp \left( -\frac{a u_{11}^2}{2uv} \right)$$

$$\cdot \exp \left( \frac{-(u_{12} - u_{11})^2}{2u(q-v)} + \frac{(u_{21} - u_{11})^2}{2v(p-u)} + \frac{(u_{22} - u_{11})^2}{2(p-u)(q-v)} \right)$$

$$\cdot du_{21} \, du_{12} \, du_{11} \, du_{u_{22}}$$

From (3.10) and (3.11), we have

$$E^x[J_a \cdot X] Z = \frac{1}{2a} \frac{1}{\sqrt{2\pi}pq} \int_{-a}^{a} \int_{-a}^{a} \left[ \frac{5}{36} (pq)^2 + \frac{pq}{9} u_{22}^2 \right] \exp \left( -\frac{u_{22}^2}{2pq} \right) \, du_{22}.$$
Conditional Yeh-Wiener integrals

\[ E^y[Z] = \int_{\mathcal{X}^{-1}(\mathcal{F})} Z(x) \, dm_y(x) = \int_{\mathcal{F}} E^y(Z \mid X) (w) \, dP_x(w) \]

\[ = \frac{1}{\sqrt{2\pi pq}} \int_{\mathcal{F}} \left[ \frac{5}{36} (pq)^2 + \frac{pq}{9} w^2 \right] \exp\left(-\frac{w^2}{2pq}\right) \, dm(w) \]

\[ = \frac{(pq)^2}{4} \]

as can be obtained by a direct computation of \( E^y[Z] \).

The following theorem is an extension of Theorem 4 in [10; p.663] for the Yeh-Wiener measure space \((C_2[Q], \mathcal{F}, m_y)\).

**Theorem 3.6.** Let \( X \) and \( Z \) be two Yeh-Wiener functionals on \( C_2[Q] \) defined by, respectively

\[ X(x) = x(p, q) \text{ and} \]

\[ Z(x) = \int \alpha_1(s, t) \, dx(s, t), \ldots, \int \alpha_n(s, t) \, dx(s, t), \quad x \in C_2[Q] \]

where \( \{\alpha_1, \alpha_2, \ldots, \alpha_n, 1\} \) is an orthonormal set in \( L^2[Q] \), and \( f \) is a real valued Lebesgue measurable function on \( R^n \) such that

\[ \langle 2\pi \rangle^{n} \prod_{i=1}^{n} \frac{1}{||\alpha_j||^2} \cdot \int_{R^n} |f(u_1, \ldots, u_n)| \exp\left(-\frac{1}{2} \sum_{j=1}^{n} \frac{u_j^2}{||\alpha_j||^2}\right) \, du_1 \ldots du_n < \infty. \]

Then we have the following:

1. \( Z \) is a real valued Yeh-Wiener integrable on functional \( C_2[Q] \).
2. There exists a version of \( E^y(Z \mid X) \) such that \( E^y(Z \mid X)(w) = M \) for \( w \in R^1 \) where \( M = E^y[Z] \) given by (2.2).

**Proof.** (1) From (3.13) and (2.2), we have \( E^y[|Z|] < \infty \), and hence \( Z \) is real valued Yeh-Wiener integrable functional on \( C_2[Q] \) and its integral is given by the right-hand side of (2.2).

(2) Since \( \int Q 1 \, dx = x(p, q) = X(x) \) for \( m_y \)-a.e. \( x \in C_2[Q] \), we have

\[ E^y[e^{iuXZ}] = E^y \left[ \exp\left(\int_Q 1 \, dx \right) f \left( \int_Q \alpha_1 \, dx, \ldots, \int_Q \alpha_n \, dx \right) \right]. \]

Applying (2.2) to the function \( h \) defined by \( h(u_1, \ldots, u_n, w) e^{iuw} f(u_1, \ldots, u_n) \) for \( (u_1, \ldots, u_n, w) \in R^{n+1} \), we have

\[ E^y[e^{iuXZ}] = M \cdot \frac{1}{\sqrt{2\pi pq}} \int_{-\infty}^{\infty} \exp(iuw) \exp\left(-\frac{w^2}{2pq}\right) \, dw \]
\[ = M \cdot \exp \left( -\frac{b q u^2}{2} \right), \quad u \in \mathbb{R}^1. \]

Since \( \int_{\mathbb{R}^1} |E_y[e^{i u X}]| du < \infty \), by Theorem 3.5 it follows that there exists a version of \( E_y(Z | X) \frac{dP_x}{dm} \) such that

\[
E_y(Z | X)(w) \frac{dP_x}{dm}(w) = M \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-iuw) \exp\left(-\frac{b q u^2}{2}\right) du
\]

\[
= \frac{M}{\sqrt{2\pi b q}} \exp\left(-\frac{w^2}{2bq}\right).
\]

Hence by (3.4), we have

\[ E_y(Z | X)(w) = M \text{ for } w \in \mathbb{R}^1. \]

As examples of application of Theorem 3.6 in evaluating conditional Yeh–Wiener integrals we have the following:

**Example 3.** Let \( X \) and \( Z \) be Yeh–Wiener functionals on \( C_2[Q] \) defined by, respectively

\[ X(x) = x(p, q) \quad \text{and} \quad Z(x) = \int_{Q} \alpha d \alpha \quad \text{for} \quad x \in C_2[Q], \]

where \( \{\alpha, 1\} \) is an orthogonal set in \( L^2[Q] \) and \( f \) is a function on \( \mathbb{R}^1 \) defined by \( f(u) = u^n \) where \( n \) is a natural number. Then we have

\[
\frac{1}{\sqrt{2\pi}||\alpha||} \int_{\mathbb{R}^1} |u|^n \exp\left\{-\frac{1}{2} \frac{u^2}{||\alpha||^2} \right\} du = \left\{ \frac{2^n}{\pi} \right\} \frac{1}{2} \cdot ||\alpha||^n \Gamma\left( \frac{n+1}{2} \right) < \infty
\]

where \( \Gamma \) is the gamma function. Hence \( E_y[|Z|] < \infty \); i.e. \( Z \) is Yeh–Wiener integrable and its integral \( E_y[Z] \) is given by

\[ E_y[Z] = M_n = \begin{cases} 0 & \text{if } n=2k-1 \\ 1 \cdot 3 \cdots (2k-1) ||\alpha||^{2k} & \text{if } n=2k \end{cases} \]

Thus we have a version of \( E_y(Z | X) \) such that \( E_y(Z | X)(w) = M_n \) for \( w \in \mathbb{R}^1 \).

**Example 4.** Let \( X \) and \( Z \) be two functionals on \( C_2[Q] \) as in Example 3 with \( f \) on \( \mathbb{R}^1 \) defined by \( f(u) = \exp(\xi u) \), where \( \xi \in \mathbb{R}^1 \). Then we have

\[
\frac{1}{\sqrt{2\pi}||\alpha||} \int_{\mathbb{R}^1} |\exp(\xi u)| \exp\left\{-\frac{1}{2} \frac{u^2}{||\alpha||^2} \right\} du = \exp\left\{ \frac{\xi^2}{2} ||\alpha||^2 \right\} < \infty.
\]

Hence \( E_y[|Z|] < \infty \) and \( E_y[Z] \) is given by \( \exp\left\{ \frac{\xi^2}{2} ||\alpha||^2 \right\} \). Thus we have a version of \( E_y(Z | X) \) such that \( E_y(Z | X)(w) = \exp\left\{ \frac{\xi^2}{2} ||\alpha||^2 \right\} \) for \( w \in \mathbb{R}^1 \).
References


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