ON EIGEN-FORMS ON SURFACES WITH NULL GAUSSIAN CURVATURE IN ELLIPTIC SPACE $S_3$

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1. Introduction

Let $(F, g)$ be an oriented compact connected $n$-dimensional Riemannian manifold. To each $p$-form $\omega$ on $F$, there is associated the $(p+1)$-form $d\omega$ and the $(n-p)$-form $*\omega$ respectively, $*$ being the Hodge operator.

The exterior codifferential $\delta$ is then defined by

$$\delta \omega = (-1)^p *^{-1} D*\omega,$$

$*^{-1}$ being the inverse mapping to $*([2], [3], [4])$. The Laplacian $\Delta$ on $p$-forms is given by

$$\Delta \omega = (D\delta + \delta D)\omega.$$  

We say that $\lambda \in \mathbb{R}$ belongs to $\text{spec}^{(p)}(\Delta)$ if there is a nontrivial $p$-form $\omega$ on $F$ such that

$$\Delta \omega = \lambda \omega.$$ (3)

The general problem is to exhibit $\text{spec}^{(p)}(\Delta)$ for a given $(F, g)$. Up to now, little is known. $\text{spec}^{(0)}(\Delta)$ is known just for the hypersphere. Recently [5] it has been proved that for a unit sphere of the Euclidean space $E^3$ $\text{spec}^{(1)}(\Delta)$ equal to 2. There are no general methods for solving the general problem. We are going to use the Stokes theorem

$$\int_F D\phi = 0$$ (4)

where $\phi$ is an $(n-1)$ form.

Consider an 3-dimensional projective space $P_3$ referred to a moving frame $\{A_i\}$ of four linearly independent analytic points $A_1$, $A_2$, $A_3$, $A_4$. An infinitesimal displacement of such a frame is determined by the equations.

$$dA_i = \omega^j_i A_j, \ (i, j, k = 1, 2, 3, 4)$$ (5)

where the one-forms $\omega^j_i$ (Pfaff's differential forms) are invariant one-forms of the projective group $PG(3, \mathbb{R})$ whose structural equations have the form
A homogeneous space $S_3=(P_3, H^3_1)$ is called an elliptic space if $H^3_1$ is a subgroup of the group $PG(3, R)$, the transformations in the subgroup $H^3_1$ do not move a non-degenerate imaginary quadric (absolute) $\sigma$. We choose a moving frame conjugate to any arbitrary manifold embedded in $S_3$ as a normalized polar tetrahedron $\{A_j\}$. In such moving frame, the absolute $\sigma$ is determined by the equation

$$\sum_{i=1}^{4} (x_i^2)^2 = 0. \quad (7)$$

The conditions of the stationary subgroup $H^3_1$ are

$$\omega^i_1 = 0, \quad \omega^i_j + \omega^j_i = 0 \quad (8)$$

2. Linear forms on surface

Let $F$ be a closed surface with null Gaussian Curvature. We are going to investigate its coordinate neighbourhood $U\subset F$. To each point $A_1 \in F$, let us associate a moving normalized polar tetrahedron $\{A_j\}$ such that the points $A_2$, $A_3$ are in the tangent plane to the surface $F$ at the point $A_1$.

The fundamental equations of a moving tetrahedron are:

\[
\begin{align*}
    dA_1 &= \omega^2_1 A_2 + \omega^3_1 A_3, \quad \omega^4_1 = 0, \\
    dA_2 &= \omega^1_2 A_1 + \omega^3_2 A_3 + \omega^4_2 A_4, \\
    dA_3 &= \omega^1_3 A_1 + \omega^2_3 A_2 + \omega^4_3 A_4, \\
    dA_4 &= \omega^2_4 A_2 + \omega^3_4 A_3.
\end{align*}
\]

(9)

The differential equation of the surface $F$ in the first differential neighbourhood is

$$\omega^4_1 = 0. \quad (10)$$

Exterior differentiation and using Cartan's Lemma [4] we get,

$$\omega^4_2 = \alpha \omega^2_1 + \beta \omega^3_1,$$

$$\omega^4_3 = \beta \omega^2_1 + \gamma \omega^3_1. \quad (11)$$

The Gaussian curvature of the surface $F$ is given by

\[
D\omega^j_i = \omega^k_i \wedge \omega^j_k. \quad (6)
\]
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\[ K = \frac{D\omega_3}{\omega_1^2 \Delta \omega_1^3} = \frac{\omega_3^4 \Delta \omega_1^2 + \omega_3^4 \Delta \omega_2^2}{\omega_1^2 \Delta \omega_1^3} \]

\[ = 1 + \alpha \gamma - \beta^2. \]  

(12)

Hence the differential equations of the surface \( F \) in the second differential neighbourhood are

\[ \omega_1^4 = 0, \]
\[ \omega_2^4 = \alpha \omega_1^2 + \beta \omega_1^3, \]
\[ \omega_3^4 = \beta \omega_1^2 + \gamma \omega_1^3, \]

(13)

with

\[ 1 + \alpha \gamma - \beta^2 = 0. \]

The purpose of this work is to prove the following

THEOREM. Let \( (F, g) \) be a closed surface with null Gaussian curvature in elliptic space \( S^3 \), \( g \) being the induced metric. Let \( \lambda \in \text{spec}(\lambda) \). Then the most general eigenvalue satisfying \( \Delta \omega = \lambda \omega \) is that \( \lambda = 0 \).

PROOF. On the surface \( F \), be given a 1-form \( \omega \) in \( U \).

\[ \omega = a \omega_1^2 + b \omega_1^3, \]  

(14)

\( a, b : U \rightarrow \mathbb{R} \) being functions. They are defined by

\[ da - b \omega_2^3 = a_1 \omega_1^2 + a_2 \omega_1^3, \]
\[ db + a \omega_3^3 = b_1 \omega_1^2 + b_2 \omega_1^3. \]

(15)

The exterior differentiation implies

\[ \left\{ da_1 - (a_2 + b_1) \omega_2^3 \right\} \Delta \omega_1^2 + \left\{ da_2 + (a_1 - b_2) \omega_2^3 \right\} \Delta \omega_1^3 = 0. \]

(16)

\[ \left\{ db_1 + (a_1 - b_2) \omega_3^3 \right\} \Delta \omega_1^2 + \left\{ db_2 + (a_2 + b_1) \omega_3^3 \right\} \Delta \omega_1^3 = 0. \]

Applying here Cartan’s lemma we get the functions.

\( a_{ij}, b_{kj} : U \rightarrow \mathbb{R} \) such that

\[ da_1 - (a_2 + b_1) \omega_2^3 = a_{11} \omega_1^2 + a_{12} \omega_1^3, \]
\[ da_2 + (a_1 - b_2) \omega_2^3 = a_{21} \omega_1^2 + a_{22} \omega_1^3, \]
\[ db_1 + (a_1 - b_2) \omega_3^3 = b_{11} \omega_1^2 + b_{12} \omega_1^3, \]
\[ db_2 + (a_2 + b_1) \omega_3^3 = b_{21} \omega_1^2 + b_{22} \omega_1^3. \]

(17)
The consequences of exterior differentiation of (17) are
\[
\begin{align*}
\{ da_{11} - (2a_{12} + b_{11})\omega_3 \} \land \omega_1 &= (da_{12} - (a_{11} - a_{22} - b_{12})\omega_3) \land \omega_1 = 0, \\
\{ da_{12} + (a_{11} - a_{22} - b_{12})\omega_3 \} \land \omega_1 &= (da_{22} + (2a_{12} - b_{22})\omega_3) \land \omega_1 = 0, \\
\{ db_{11} - (a_{11} - 2b_{12})\omega_3 \} \land \omega_1 &= (db_{12} + (b_{12} + b_{11} - b_{22})\omega_1) \land \omega_1 = 0, \\
\{ db_{12} + (a_{12} + b_{11} - b_{22})\omega_3 \} \land \omega_1 &= (db_{22} + (a_{22} + 2b_{12})\omega_3) \land \omega_1 = 0. 
\end{align*}
\]
(18)

Using Cartan's lemma equations (18) give the existence of functions
\[ A_1, B_1: = U \longrightarrow \mathbb{R} \text{ such that} \]
\[
\begin{align*}
da_{11} - (2a_{12} + b_{11})\omega_3 &= A_1\omega_1 + A_2\omega_3, \\
da_{12} + (a_{11} - a_{22} - b_{12})\omega_3 &= A_2\omega_1 + A_3\omega_3, \\
da_{22} + (2a_{12} - b_{22})\omega_3 &= A_4\omega_1 + A_4\omega_3, \\
db_{11} - (a_{11} - 2b_{12})\omega_3 &= B_1\omega_1 + B_2\omega_3, \\
db_{12} + (a_{12} + b_{11} - b_{22})\omega_3 &= B_2\omega_1 + B_3\omega_3, \\
db_{22} + (a_{22} + 2b_{12})\omega_3 &= B_4\omega_1 + B_4\omega_3. 
\end{align*}
\]
(19)

Now for 1-form, we have
\[
\begin{align*}
* (p\omega_1 + q\omega_3) &= -q\omega_1 + p\omega_1, \\
*^{-1} (p\omega_1 + q\omega_3) &= q\omega_1 - p\omega_1. 
\end{align*}
\]
(19) (20)

\[ D\omega = (\ast^{-1} D\ast D - D\ast^{-1} D\ast)\omega. \]
(21)

In our case we have
\[ \omega = a\omega_1^2 + b\omega_3^3. \]
\[ D\omega = (b_1 - a_2)\omega_1^2 \land \omega_3^3, \]
\[ \ast D\omega = b_1 - a_2. \]
\[ D\ast D\omega = (b_{11} - a_{12})\omega_1^2 + (b_{12} - a_{22})\omega_3^3, \]
\[ \ast^{-1} D\ast D\omega = (b_{12} - a_{22})\omega_1^2 - (b_{11} - a_{12})\omega_1^2, \]
\[ \ast\omega = -b\omega_1^2 + a\omega_3^3, \]
\[ D\ast\omega = (b_2 + a_1)\omega_1^2 \land \omega_3^3, \]
\[ \ast^{-1} D\ast\omega = b_2 + a_1. \]
(22)
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\[ D^*^{-1} D \omega = (a_{11} + b_{12}) \omega_1^2 + (a_{12} + b_{22}) \omega_1^3. \]  

(23)

Hence for the 1-form \( \omega \) the Laplacian

\[ \Delta \omega = -(a_{11} + a_{22}) \omega_1^2 - (b_{11} + b_{22}) \omega_1^3. \]  

(24)

If the form \( \omega \) satisfying (3), then

\[ a_{11} + a_{22} = -\lambda a, \quad b_{11} + b_{22} = -\lambda b. \]  

(25)

Because of equations (19) we get

\[ A_1 + A_3 = -\lambda a_1, \quad A_2 + A_4 = -\lambda a_2, \]
\[ B_1 + B_3 = -\lambda b_1, \quad B_2 + B_4 = -\lambda b_2. \]  

(26)

For a general form \( \omega \) we can get

\[ D^* D [(a_1 - b_2)^2 + (a_2 + b_1)^2] = 2[(a_{11} - b_{12})^2 + (a_{12} - b_{22})^2 +
+ (a_{12} + b_{11})^2 + (a_{22} + b_{12})^2] \omega_1^2 \wedge \omega_1^3
+ (-2\lambda) [(a_1 - b_2)^2 + (a_2 + b_1)^2] \omega_1^2 \wedge \omega_1^3. \]

Using the stockes theorem on \( D^* D \omega \), we get

\[ a_1 - b_2 = 0, \quad a_3 + b_1 = 0, \]
\[ a_{11} - b_{12} = a_{12} - b_{22} = a_{12} + b_{11} = a_{22} + b_{12} = 0. \]

From which follows that

\[ a_{11} + a_{22} = 0, \quad b_{11} + b_{22} = 0. \]  

(27)

Comparing (27) with (25) it follows directly that

\[ \lambda = 0. \]

This proves our theorem.

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REFERENCES