In [8] G. Sampson and Z. Zielezny studied the hypoelliptic convolution equations in the space of distributions with exponential growth of polynomial powers. They found the structure theorems and Paley-Wiener type theorems for those distributions. In addition they established the meaning of hypoelliptic convolution equations in those spaces and classified the class of hypoelliptic convolution operators in those distribution spaces in terms of their Fourier transform.

We are going to study the same problems in the spaces of distributions with general exponential powers which was defined by convex functions in [4]. Especially in this paper we only concern about the structure theorems for those distributions and convolution operators and Paley-Wiener type theorems of those distribution and test functions.

We will study the convolution equations in those spaces later.

1. Dual functions in the sense of young

Let $\mu(\xi)$ ($0 \leq \xi \leq \infty$) denote a continuous increasing function such that $\mu(0) = 0$, $\mu(\infty) = \infty$. For $x \geq 0$ we define

$$M(x) = \int_0^x \mu(\xi) \, d\xi.$$ 

The function $M(x)$ is an increasing, convex, continuous function, with $M(0) = 0$, $M(\infty) = \infty$ and satisfies the fundamental convexity inequality $M(x_1) + M(x_2) \leq M(x_1 + x_2)$. Further we define $M(x)$ for negative $x$ by means of the equality $M(-x) = M(x)$. Note that since the derivative $\mu(x)$ of $M(x)$ is unbounded in $\mathcal{R}$, the function $M(x)$ itself will grow faster than any linear function as $|x| \rightarrow \infty$.

DEFINITION 1.1. Let $M(x)$ and $Q(y)$ be the functions corresponding to $\mu(\xi)$ and $\omega(\eta)$ as above. Then $M(x)$ and $Q(y)$ are said to be dual in the sense of Young if and only if $\mu(\xi)$ and $\omega(\eta)$ are mutually inverse, i.e. $\mu[\omega(\eta)] = \eta$ and $\omega[\mu(\xi)] = \xi$.

The following pairs of functions are examples of functions which are dual in
In the sense of Young:

\[ M(x) = x^p/p, \quad \Omega(y) = y^q/q, \quad \frac{1}{p} + \frac{1}{q} = 1; \]

\[ M(x) = e^x - x - 1, \quad \Omega(y) = (y+1)\log(y+1) - y. \]

The function \( M(x) \) can be defined on \( \mathcal{R}^n \) by \( M(x_1, \ldots, x_n) = M(x_1) + \cdots + M(x_n) \) for all \( x \in \mathcal{R}^n \).

Let us note a few properties of dual functions which will be useful in what follows.

**LEMMA 1.1.** [4] (Young's Inequality).

\[ xy \leq M(x) + \Omega(y) \text{ for any } x \geq 0, \ y \geq 0. \]

and the equality holds if and only if \( y = \mu(x) \).

**LEMMA 1.2.** [4] If \( M(x) \) is dual to \( \Omega(y) \), \( M_1(x) \) is dual to \( \Omega_1(y) \), and \( M(x) < M_1(x) \) for sufficiently large values of \( x \), then \( \Omega(y) > \Omega_1(y) \) for sufficiently large values of \( y \).

The next lemma will be used repeatedly in the following sections and chapters.

**LEMMA 1.3.** Let \( M(x) \) and \( \Omega(y) \) corresponding to \( \mu(\xi) \) and \( \omega(\xi) \), respectively, and dual each other.

Then

\[ \sup_{x \in \mathcal{R}^n} e^{-M(kx) + \langle x \eta \rangle} = e^{\frac{1}{k} \eta}; \]

\[ \sup_{x \in \mathcal{R}^n} e^{-\Omega(kx) + \langle x \eta \rangle} = e^{M\left(\frac{1}{k} \eta\right)} \]

for all \( k \in \mathcal{R} \) and \( k < 0 \) where \( \langle x \eta \rangle = x_1 \eta_1 + \cdots + x_n \eta_n \).

**PROOF.** It is sufficient to show that the results hold for \( n=1 \), because of the definition of the function \( M(x) \) in \( \mathcal{R}^n \). Say \( f(t) = -M(kt) + t\eta \), then we can get the maximum point \( t = \frac{1}{k} \omega\left(\frac{1}{k} \eta\right) \) of \( f(t) \) by solving the equation \( f'(t) = -k\mu(kt) + \eta = 0 \). Then the maximum value of \( f(t) \) is \( -M\left(\omega\left(\frac{1}{k} \eta\right)\right) + \frac{1}{k} \omega\left(\frac{1}{k} \eta\right) |\eta| = \Omega\left(\frac{1}{k} \eta\right) \) by Young's inequality. Therefore, we have

\[ \sup_{t \in \mathcal{R}} e^{-M(kt) + t\eta} = e^{\Omega\left(\frac{1}{k} \eta\right)}. \]

Now we list some properties of these functions which will be frequently used in the later chapters.
(1.1) $M(x) + M(y) \leq M(x+y)$ for all $xy \geq 0$.
(1.2) $M(x) \leq x M(x)$ for all $x > 0$.
(1.3) $M(\frac{1}{k}x) \leq \frac{1}{k} M(x)$ for any $k > 0$.
(1.4) $M(x+y) \leq M(2x) + M(2y)$ for all $xy \geq 0$.


Using the function $M(x)$ which we introduced in the previous section, we define the space $K_M$ as the space of all functions $\varphi(x)$ in $C^\infty(\mathbb{R}^n)$ such that

$$\nu_k(\varphi) = \sup_{x \in \mathbb{R}^n} e^{M(k|x|)} |D^\alpha \varphi(x)| < \infty, \ k = 1, 2, \ldots$$

where $D^\alpha = (i^{\frac{-1}{\alpha_1}} \frac{\partial}{\partial x_1})^{\alpha_1} \ldots (i^{\frac{-1}{\alpha_n}} \frac{\partial}{\partial x_n})^{\alpha_n}$ and $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$.

The space $K_M$ with $M(x) = |x|^p$ was discussed in [8] and similar spaces, not exactly the same, were used in [4] to solve the Cauchy problem. The topology in $K_M$ is defined by the countable family of semi-norms $\{\nu_k\}_{k=1}^\infty$. Then the space $K_M$ became a Frechet space (see Th. 1.2.1 below) and the identity mappings $\mathcal{D}(\mathbb{R}^n) \rightarrow K_M \rightarrow \mathcal{E}(\mathbb{R}^n)$ are continuous, where $\mathcal{E}$ denotes the space of all $C^\infty$-functions and $\mathcal{D}$ the space of $C^\infty$-functions with compact support in $\mathbb{R}^n$.

**Theorem 2.1.** The space $K_M$ with semi-norms $\nu_k, 1, 2, \ldots$ is a Frechet space.

**Proof.** It is sufficient to show that the space $K_M$ is complete. Let $\{\varphi_j\}$ be a Cauchy sequence in $K_M$, then the sequence $\{\varphi_j\}$ is obviously a Cauchy sequence in $C^\infty(\mathbb{R}^n)$, and so the completeness of $C^\infty(\mathbb{R}^n)$ implies that there is a function $\varphi$ in $C^\infty(\mathbb{R}^n)$ such that $\varphi_j$ converges to $\varphi$ in $\mathcal{E}$. Since the sequence $\{\varphi_j\}$ is Cauchy in $K_M$, $\nu_k(\varphi_j) \leq C_k < \infty$ for every $j = 1, 2, \ldots$ and $k = 1, 2, \ldots$ Since the sequence $\varphi_n$ converges to $\varphi$ in $C^\infty(\mathbb{R}^n)$, given any $\varepsilon > 0$ we obtain

$$e^{M(k|x|)} |D^\alpha \varphi(x)| \leq e^{M(k|x|)} |D^\alpha \varphi_j(x)| + \varepsilon$$

$$\leq \nu_k(\varphi_j) + \varepsilon \leq C_k + \varepsilon < \infty$$

for sufficiently large $j$ and all $x \in \mathbb{R}^n$. Therefore, $\nu_k(\varphi) \leq C_k + \varepsilon < \infty$ for every $k = 1, 2, \ldots$. This implies that $\varphi \in K_M$. For the proof that the sequence $\varphi_j$ converges to $\varphi$, we consider the sequence $[(\varphi_i - \varphi) - (\varphi_j - \varphi)]$ in $i, j = 1, 2, \ldots$. Since $\{\varphi_n\}$ is a Cauchy sequence in $K_M$, for given $\varepsilon > 0$ and $k = 1, 2, \ldots$, there exists
an integer \( n_0 > 0 \) such that \( \nu_k(\varphi_i - \varphi_j) < \varepsilon \) for all \( i, j \geq n_0 \). Also we know that the sequence \( \{(\varphi_i - \varphi) - (\varphi_j - \varphi)\} \) for all fixed \( i > n_0 \) converges to \( \varphi_i - \varphi \) in \( K_{\mathcal{M}} \) as \( j \to 0 \), and this implies that

\[
\nu_k(\varphi_i - \varphi) \leq \nu_k((\varphi_i - \varphi) - (\varphi_j - \varphi)) + \varepsilon \\
= \nu_k(\varphi_i - \varphi_j) + \varepsilon < 2^\varepsilon,
\]

provided that \( i, j \geq n_0 \). Thus \( \varphi_n \) converges to \( \varphi \) in \( K_{\mathcal{M}} \).

**REMARK.** Formally one could have constructed a space \( K_{\mathcal{M}} \) starting from any non-negative continuous function \( M(x) \) without taking into account whether this function has the special form which we assumed in §1.1 (later we shall make use of this special form), by means of the definition which we made above. In this case, one may obtain the same space \( K_{\mathcal{M}_1} = K_{\mathcal{M}_2} \) for two different functions \( M_1(x) \) and \( M_2(x) \). There is a simple sufficient condition (see [4]) for this equality to hold, viz. there are positive constants \( r_1, r_2, r_3 \) such that \( M_1(r_1 x) \leq M_2(r_2 x) \leq M_3(r_3 x) \) for all sufficiently large \( x \geq 0 \). For example, we have the same space \( K_{\mathcal{M}_1} = K_{\mathcal{M}_2} \) when \( M_1(x) = e^x - x - 1 \) and \( M_2(x) = e^x M_2(x) \) does not satisfy the requirement in 1.

We denote by \( K'_{\mathcal{M}} \) the space of all continuous linear functional on \( K_{\mathcal{M}} \). The restriction \( \tilde{T} \) of a functional \( T \in K'_{\mathcal{M}} \) to \( \mathcal{D} \) is a distribution. Also, since \( \mathcal{D} \) is dense in \( K_{\mathcal{M}} \), \( T \) is uniquely determined by its values on \( \mathcal{D} \), that is, by \( \tilde{T} \). Thus we can identify \( T \) with \( \tilde{T} \) and regard \( K'_{\mathcal{M}} \) as a subspace of the space \( \mathcal{D}'(\mathbb{R}^n) \) of distributions.

In order to characterize the structure of distributions in \( K'_{\mathcal{M}} \) by their “growth” at infinity, we need the following result.

**LEMMA 1.2** [9 theorem XXII]. If \( B' \) is a bounded set of distributions in \( \mathcal{D}' \), then for every relative compact open set \( \Omega \) in \( \mathbb{R}^n \) there exists an integer \( m \geq 0 \) such that, for all \( \varphi \in \mathcal{D}' \) whose support is contained in a neighborhood of origin in \( \mathbb{R}^n \), the set \( T*\varphi, T \in B' \), of continuous functions is uniformly bounded in \( \Omega \).

Using this lemma, we prove the structure theorem.

**THEOREM 2.3.** A distribution \( T \) in \( \mathcal{D}' \) is in \( K'_{\mathcal{M}} \) if and only if there exist positive integers \( m, k \) and a bounded continuous function \( f(x) \) on \( \mathbb{R}^n \) such that

\[
T = \frac{\partial^m}{\partial x_1^m \cdots \partial x_n^m} [e^{M(kx)} f(x)].
\]
**Proof.** It is obvious that a distribution $T$ of the form (1) is in $K'_M$.

Conversely, suppose that $T$ is in $K'_M$. First we show that there is a positive integer $k_0$ such that the set of distributions

$$
\{ e^{-M(k,y)} \tau_i T_x : y \in \mathcal{D}^n \}
$$

where $\tau_i T_x$ is the translation of $T_x$ by $y$, is bounded in $\mathcal{D}'$.

Since $T$ is continuous in $K_M$ and semi-norms $\nu_k$ are increasing, there exist $\epsilon > 0$ and a positive integer $k_1$ such that

$$
\nu_k(\varphi) \leq \epsilon \implies |T(\varphi)| \leq 1
$$

for all integers $k \geq k_1$ and all $\varphi$ in $K_M$.

On the other hand, we have $M(x+y) \leq M(2x) + M(2y)$ by (1.4) and therefore we get

$$
\nu_k(\varphi(x+y)) = \sup_{x \in \mathcal{D}^n} e^{M(kx)} |D^\alpha \varphi(x+y)|
$$

and

$$
= \sup_{x \in \mathcal{D}^n} e^{M(k(x-y))} |D^\alpha \varphi(x)|
$$

$$
\leq e^{M(2ky)} \sup_{x \in \mathcal{D}^n} e^{M(2kx)} |D^\alpha \varphi(x)|
$$

$$
\leq e^{M(2ky)} \nu_{2k}(\varphi) \text{ for all } \varphi \in K_M.
$$

Consequently, if we choose $k_0 \geq 2k_1$, we have from (3) and (4) that

$$
| \langle T_x, e^{\epsilon \nu_{2k}(\varphi)} e^{-M(k,y)} \varphi(x+y) \rangle | \leq 1
$$

and then

$$
| \langle e^{-M(k,y)} \tau_i T_x, \varphi(x) \rangle |
$$

$$
= | \langle T_x, e^{-M(k,y)} \varphi(x+y) \rangle |
$$

$$
\leq \frac{\nu_{2k}(\varphi)}{\epsilon}
$$

for all $\varphi \in \mathcal{D}$, which proves that the set (2) is bounded in $\mathcal{D}'$.

According to Lemma 2.2, for every relative compact open set $\Omega$ in $\mathcal{R}^n$, there exist an integer $N \geq 0$ and a sufficiently small neighborhood $\omega$ of the origin such that, for every $\varphi \in \mathcal{D}^N(\omega)$, the set $\{ e^{-M(k,y)} \tau_i (T \ast \varphi) : y \in \mathcal{R}^n \}$ is a bounded set of continuous functions in $\Omega$. It follows that $e^{-M(k,x)}(T \ast \varphi)(x)$ is a bounded, continuous function in $\mathcal{R}^n$. 
Let now $E$ be a fundamental solution for the iterated Laplace operator $\Delta^m$ (for the existence of such solution $E$, see [7]), i.e. $\Delta^m E = \delta$. If $m$ is sufficiently large, $E$ is $N$-times continuously differentiable and $E \in C^\infty(\mathbb{R}^n \setminus \{0\})$. Therefore, if $\varphi \in \mathcal{D}(\omega)$ with $\varphi = 1$ in a neighborhood of the origin, we have $\varphi E \in \mathcal{D}^N(\omega)$ and $\delta = \Delta^m(\varphi E) - W$ where $W \in \mathcal{D}(\omega)$. Hence $T = \Delta^m(\varphi E \ast T) - W \ast T$ and so

$$T = \sum_{|\alpha| \leq m} D^\alpha \{e^{M(k,x)} f_\alpha(x)\}$$

where $f_\alpha$ are bounded continuous functions in $\mathcal{B}^n$, since $\varphi E$ and $W$ are in $\mathcal{D}^N(\omega)$. Taking primitive functions, if necessary, one can reduce the right hand side of (5) to one single term of the form (1).

As usual, we introduce in $K'_M$ the topology of uniform convergence on all bounded sets in $K_M$.

3. Convolutions in $K'_M$.

The convolution of a distribution $T$ in $K'_M$ and a function $\varphi$ in $K_M$ is defined by

$$(T \ast \varphi)(x) = (\varphi \ast T)(x) = T \ast \varphi(x-y)$$

Throughout this paper we will use the same symbol $C$ representing constants even though they will differ from one another in different steps of estimate.

**Lemma 3.1.** Let $T$ be a distribution in $K'_M$ and $\varphi$ a function in $K_M$. Then $T \ast \varphi$ is a $C^\infty$-function in $\mathbb{R}^n$ such that, for some integer $k_0 \geq 0$,

$$|D^\alpha(T \ast \varphi)(x)| \leq C_\alpha e^{M(k,x)}$$

for every $\alpha \in \mathbb{N}^n$.

where $C_\alpha$ are constants depending on $\alpha$ (such function is an element in $\mathcal{D}'K'_M$ which we will define later).

**Proof.** By theorem 2.3 $T = D^m f$ for some integer $m \geq 0$ where $f(x)$ is continuous function satisfying the estimation $f(x) = 0(e^{M(k,x)})$ for some integer $k \geq 0$. Therefore, for every multi-index $\alpha$, we have

$$|D^\alpha(T \ast \varphi)(x)| = |D^\alpha(D^m f \ast \varphi)(x)|$$

$$= |f \ast D^\alpha D^m \varphi(x)|$$

$$\leq \int_{\mathbb{R}^n} |f(y)| |D^\alpha D^m \varphi(x-y)| \, dy$$

$$\leq C \int_{\mathbb{R}^n} |f(y)| e^{-M(4k(x-y))} \, dy$$

$$\leq C \int_{\mathbb{R}^n} e^{M(k,y) - M(4k(x-y))} \, dy.$$
for some constant $C$, since $\phi$ is in $K_M$.

Using the fact that $M(4k(x-y))+M(4kx)\geq M(2ky)$ by (1.4) and $2M(ky)\leq M(2ky)$ by (1.1), we obtain

$$|D^\alpha(T*\phi)(x)|\leq C \int_{\mathbb{R}^n} e^{\tau(ky) - M(4k(x-y))} dy$$

for some constant $C$ where $k$ does not depend on $\alpha$. Hence if we take $k_0=4k$, the lemma follows.

Throughout this paper, we assume that the generating function $\mu(\xi)$ of $M(x)$ is a smooth function in $\mathbb{R}^n/\{0\}$. Otherwise we can approximate $\mu$ by a smooth function $\mu_1$ which has the following property For $M_1(\xi)=\int_0^1 \mu_1(\xi) d\xi$, there exist $r_1, r_2>0$ such that $M_1(r_1x)\leq M(x)\leq M_1(r_2x)$ for sufficiently large $x$. By our previous discussion, we have $K_M=K_M$. Under this assumption, $M(x)$ is a $C^\infty$-function in $\mathbb{R}^n-\{0\}$.

Let $r_k(x), k=1, 2, 3, \ldots$ be a $C^\infty$-function in $\mathbb{R}^n$ such that

$$(6) \quad r_k(x)=e^{M(kx)} \text{ for } |x|\geq 1.$$  

**THEOREM 3.2.** For a distribution $S$ in $K'_M$, the following conditions are equivalent:

(i) the distribution $S_k=r_kS, k=1, 2, \ldots$ are in $\mathbb{\mathcal{D}}'$

(ii) for every integer $k\geq 0$, there exists an integer $m\geq 0$ such that $S=\sum_{|\alpha|\leq m} D^\alpha f_\alpha$ where $f_\alpha$ are continuous functions in $\mathbb{R}^n$ whose product with $e^{M(kx)}$ are bounded.

(iii) for every $\phi$ in $K_M$, the convolution $S*\phi$ is in $K_M$.

Here $\mathbb{\mathcal{D}}'$ is the space of tempered distributions in $\mathbb{R}^n$.

**PROOF.** We are going to prove the implications (i)$\iff$(ii) and (ii)$\iff$(iii).

(i)$\iff$(ii)

Suppose that condition (i) is satisfied and let $k$ be a given integer $\geq 1$. Since $S_{k+1}$ is in $\mathbb{\mathcal{D}}'$, we can write $S_{k+1}=D^\alpha f$ where $f$ is a continuous function in $\mathbb{R}^n$ such that

$$(7) f(x)=O(1+|x|^{l}), \text{ as } |x|\to\infty.$$
for some integer \( l \geq 0 \). Hence, by the Leibniz formula

\[
S = r_{k+1}^{-1} D^\alpha f
\]

where \( f_\beta(x) = (-1)^{1-\alpha} f(x) D^\alpha r_{k+1}^{-1} (x) \)

\[
= O(e^{-M(kx)}), \quad \text{as } |x| \to \infty, \quad \text{as in view of (6) and (7)}.
\]

This shows the representation (ii).

(ii) \( \Longrightarrow \) (i)

We want to show that \( S_k = r_k S \) is in \( \delta' \) for any given integer \( k \geq 1 \). In view of (ii), for any \( h_1 > k \), \( S \) can be represented in the form \( S = \sum_{|\alpha| \leq m} D^\alpha f_\alpha \) where \( m \geq 0 \) and \( f_\alpha \) are continuous functions in \( \mathcal{S}' \) such that \( f_\alpha(x) = O(e^{-M(\beta x)}) \) as \( |x| \to \infty \).

Then

\[
S_h = r_h S
\]

\[
S_h = \sum_{|\alpha| \leq m} r_\alpha D^\alpha f_\alpha
\]

\[
= \sum_{|\alpha| \leq m} \sum_{\beta \leq \alpha} D^\alpha f_\alpha
\]

where

\[
f_{\alpha \beta}(x) = (-1)^{1-\beta} f_\alpha(x) D^\beta r_\beta
\]

\[
= O(e^{-M(x)}) \quad \text{as } |x| \to \infty,
\]

in view of the growths of \( f_\alpha(x) \) and \( r_\beta(x) \). Therefore, \( S_h \) is in \( \delta' \).

(ii) \( \Longrightarrow \) (iii)

By lemma 3.1., the convolution \( S \ast \varphi \) of \( S \in \mathcal{K}'_M \) and \( \varphi \in \mathcal{K}_M \) is a \( C^\infty \)-function in \( \mathcal{R}' \). Hence it is sufficient to study the growth of the convolution \( S \ast \varphi \). We want to show that, for any given integer \( l \geq 0 \),

\[
\sup_{x \in \mathcal{R}' \setminus \mathcal{R} \setminus \mathcal{R}'} |D^\alpha (S \ast \varphi)(x)| e^{M(lx)} < \infty.
\]

By hypothesis, for a given integer \( k \) which we will choose later, there is a positive integer \( m \) such that

\[
S = \sum_{|\alpha| \leq m} D^\alpha f_\alpha
\]

where \( f_\alpha \) are continuous functions in \( \mathcal{S}' \) with

\[
f_\alpha(x) = O(e^{-M(kx)}),
\]
as \(|x| \to \infty\). For any multi-index \(\beta\),
\[
e^{M(|x|)} |D^\beta (T \ast \varphi)(x)|
\]
\[
\leq e^{M(|x|)} \sum_{|\alpha| \leq m} |D^\beta (D^\alpha f \ast \varphi)(x)|
\]
\[
= \sum_{|\alpha| \leq m} e^{M(|x|)} |f \ast D^\alpha \varphi(x)|
\]
\[
\leq \sum_{|\alpha| \leq m} e^{M(|x|)} \int_{\mathbb{R}^n} |f_\alpha(y)||D^\alpha \varphi(x-y)|dy,
\]
\[
\leq \sum_{|\alpha| \leq m} e^{M(|x|)} |f_\alpha(y)||e^{M(2l(x-y))} D^\alpha \varphi(x-y)|dy,
\]

since
\[
M(|x|) \leq M(2l(x-y)) + M(2l y).
\]

But
\[
M(2l+1)y - M(y) \geq M(2l y) \text{ by (1.1),}
\]

so that
\[
e^{M(|x|)} |D^\beta (T \ast \varphi)(x)|
\]
\[
\leq \sum_{|\alpha| \leq m} \sup_{y \in \mathbb{R}^n} \{e^{M(2l+1)|x|} |f_\alpha(y)| \} \cdot \sup_{x \in \mathbb{R}^n} \{e^{M(2l|x|)} |D^\alpha \varphi(x)| \}\int_{\mathbb{R}^n} e^{-M(y)} dy
\]
\[
\leq \infty,
\]

provided that we choose \(k\) greater than \(2l + 1\). Hence the convolution \(T \ast \varphi\) is in \(K_M\).

(iii) \(\implies\) (ii).

For any given integer \(k \geq 0\), we can see that the set of distributions \(\{e^{M(kx)} T_x S_y : x \in \mathbb{R}^n\}\) is bounded in \(\mathcal{D}'\). In fact, \(\langle T_x S_y, \varphi(y) \rangle = (S \ast \varphi')(y-x)\) where \(\varphi \in \mathcal{D}\) and \(\varphi'(x) = \varphi(-x)\). But \(S \ast \varphi' \in K_M\), and so
\[
\langle e^{M(kx)} T_x S_y, \varphi(y) \rangle = e^{M(kx)} \langle Z_x S_y, \varphi(y) \rangle
\]
\[
= e^{M(kx)} |S \ast \varphi'(-x)|
\]
\[
< \infty.
\]

Hence
\[
\{e^{M(kx)} Z_x S_y : x \in \mathbb{R}^n\}
\]
is bounded in \(\mathcal{D}'\).

By Lemma 2.2. for every relatively compact open set \(\Omega\) in \(\mathbb{R}^n\) there exist an integer \(N \geq 0\) and a sufficiently small neighborhood \(\omega\) of the origin such that,
for every \( \varphi \in \mathcal{D}^N(\omega) \), the set \( \{ e^{M(kx)} T_x(S*\varphi) : x \in \mathbb{R}^n \} \) is a set of continuous functions uniformly bounded in \( \Omega \). Hence it follows that \( e^{M(kx)} (S*\varphi)(x) \) is a bounded, continuous function in \( \mathbb{R}^n \).

Let now \( E \) be a fundamental solution for the iterated Laplace operator \( \Delta^m \). If \( m \) is sufficiently large, \( E \) is \( N \) times continuously differentiable and \( E \in C^\infty(\mathbb{R}^n-\{0\}) \). Therefore, if \( \varphi \in \mathcal{D}(\omega) \) with \( \varphi = 1 \) in a neighborhood of the origin, we have \( \varphi E \in \mathcal{D}^N(\omega) \) and \( \delta = \Delta^m(\varphi E) - W \) where \( W \in \mathcal{D}(\omega) \). Hence \( T = \Delta^m(\varphi E* \psi) - W* T \) and so

\[
T = \sum_{|\alpha| \leq m} D^\alpha (f_\alpha(x))
\]

where \( f_\alpha \) are continuous functions in \( \mathbb{R}^n \) whose product with \( e^{M(kx)} \) are bounded.

This proves (ii).

We denote by \( O'(K'_M, K'_M) \) the space of all distributions \( S \) satisfying the equivalent conditions (i)–(iii) in Theorem 3.2. In view of (iii), it is the space of convolution operators in \( K'_M \).

For \( S \in O'(K'_M, K'_M) \) and \( T \in K'_M \), we define

\[
\langle S*T, \varphi \rangle = \langle T*S, \varphi \rangle = \langle T, S*\varphi \rangle
\]

where \( \varphi \in K_M \) and \( \langle S*, \varphi \rangle = \langle S, \varphi \rangle \).

The convolution \( S*T \) is well-defined, since

\[
S*\varphi = (S*\varphi)^* \in K_M.
\]

Moreover, from the proof of the implication (ii) \( \Rightarrow \) (iii), we can see that the mapping \( \varphi \rightarrow S* \varphi \) of \( K_M \) into \( K_M \) is continuous.

**Corollary 3.3.** If \( S \) and \( T \) are distributions in \( O'(K'_M, K'_M) \), then the convolution \( S*T \) is in \( O'(K'_M, K'_M) \).

**Proof.** By theorem 3.2, for a given positive integer \( k_0 \), which we will choose later, there exist integers \( m_1 \) and \( m_2 \geq 0 \) such that

\[
S = \sum_{|\alpha| \leq m_1} D^\alpha f_\alpha
\]

and

\[
T = \sum_{|\beta| \leq m_2} D^\beta g_\beta
\]

where \( f_\alpha(x) \) and \( g_\beta(x) \) are continuous in \( \mathbb{R}^n \) and

\[
f_\alpha(x), g_\beta(x) = O(e^{-M(kx)})
\]
as $|x| \to \infty$. Since

$$S \ast T = \sum_{|\alpha + \beta| \leq m_1 + m_2} D^\alpha \ast D^\beta (f_{\alpha} \ast g_{\beta})$$

it is sufficient to show that, for any given integer $k \geq 0$, we can choose an integer $k_0 \geq 0$ such that

$$f_{\alpha} \ast g_{\beta}(x) = O(e^{-M(kx)})$$

as $|x| \to \infty$. From the fact

$$M((2k+1)y) \geq M(2ky) + M(y),$$

we have the following

$$|e^{M(kx)}(f_{\alpha} \ast g_{\beta})(x)|$$

$$\leq e^{M(kx)} \int_{\mathbb{R}^*} |f_{\alpha}(y)| |g_{\beta}(x-y)| dy$$

$$\leq \int_{\mathbb{R}^*} e^{M(2ky)} |f_{\alpha}(y)| e^{M(2k(y-x))} |g_{\beta}(x-y)| dy$$

$$\leq \int_{\mathbb{R}^*} e^{-M(y)} dy \sup_{y \in \mathbb{R}^*} |e^{M(2ky)} |f_{\alpha}(y)|| \sup_{y \in \mathbb{R}^*} |e^{M(2kx)} |g_{\beta}(x)||$$

$$\leq e^{M(kx)}$$

provided that we choose $k_0$ greater than $2k+1$. This shows the convolution $S \ast T$ is in $O'(K'_M, K'_M)$ by the theorem 3.2.

4. Fourier Transforms.

For a function $\varphi \in K_M$, the Fourier transform $\hat{\varphi}$ is defined by

$$\hat{\varphi}(\xi) = \int_{\mathbb{R}^*} e^{-i\langle x, \xi \rangle} \varphi(x) dx$$

where

$$\langle x, \xi \rangle = x_{1} \xi_{1} + x_{2} \xi_{2} + \cdots + x_{n} \xi_{n}.$$ 

Since $K_M$ is a subspace of $\delta$ the inversion formula holds, that is,

$$\varphi(x) = \frac{1}{(2\pi)^{n}} \int_{\mathbb{R}^*} e^{i\langle x, \xi \rangle} \hat{\varphi}(\xi) d\xi.$$ 

Since a distribution $S$ in $O'(K'_M, K'_M)$ is in $\delta'$, its Fourier transform $\hat{S}$ is
defined by the Parseval's equality as follows:
\[ \langle \hat{S}, \varphi \rangle = \langle S, \hat{\varphi} \rangle \text{ for every } \varphi \in \mathcal{D}. \]

Now we are going to establish a Paley-Wiener type theorem for the spaces \( K_M \) and \( O'(K'_M, K'_M) \). It is based on the theorems for the Fourier Transforms of functions in \( W_M \) (see [4]).

**THEOREM 4.1.** (a) An entire function \( F(\zeta) \) is a Fourier transform of a function \( \phi \) in \( K_M \) if and only if, for every integer \( N \geq 0 \) and every \( \varepsilon > 0 \) there exists a constant \( C \) such that
\[ |F(\xi + i\eta)| \leq C(1 + |\xi|)^{-N} e^{Q(\eta)} \]
where
\[ \zeta = \xi + i\eta \in \mathbb{C}^n. \]

(b) An entire function \( F(\zeta) \) is a Fourier transform of a distribution \( S \) in \( O'(K'_M, K'_M) \) if and only if for every \( \varepsilon > 0 \) there exist constants \( N \) and \( C \) such that
\[ |F(\xi + i\eta)| \leq C(1 + |\xi|)^{N} e^{Q(\eta)} \]
where
\[ \zeta = \xi + i\eta \text{ in } \mathbb{C}^n. \]

**PROOF.** (a) First we are going to show that the Fourier transform of \( \varphi \in K_M \) can be continued in \( \mathbb{C}^n \) as an entire function. For \( \zeta = \xi + i\eta \), we have
\begin{align*}
|\hat{\varphi}(\zeta)| &= |\int_{\mathbb{R}^n} e^{-i\langle \xi, \eta \rangle} \varphi(x) dx| \\
&\leq \int_{\mathbb{R}^n} |\varphi(x)| e^{\langle \xi, \eta \rangle} dx \\
&\leq C \int_{\mathbb{R}^n} e^{-M(2x) + \langle \eta \rangle} dx \\
&\leq C \sup_{s \in \mathbb{R}^n} \left\{ e^{-M(x) + \langle \eta \rangle} \right\} \\
&\int_{\mathbb{R}^n} e^{-M(x)} dx \\
&\leq Ce^{Q(\eta)} < \infty, \text{ by the lemma 1.3.}
\end{align*}

Hence
\[
\int_{\mathbb{R}^n} e^{-i\langle x, \xi \rangle} \varphi(x) \, dx
\]
is absolutely convergent for each \( \xi \in \mathbb{C}^n \) and so \( \hat{\varphi} \) can be continued in \( \mathbb{C}^n \). Furthermore, the function \( \hat{\varphi}(\xi) \) is differentiable (indeed, indefinitely) for every \( \xi \) in \( \mathbb{C}^n \). Indeed, after a formal differentiation with respect to \( \xi \), i.e.

\[
D^\alpha \hat{\varphi}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x, \xi \rangle} x^\alpha \varphi(x) \, dx,
\]

the integral in the right side still remains absolutely convergent because of \( x^\alpha \varphi(x) \in K_M \). This guarantees the existence of the derivatives of the function \( \hat{\varphi}(\xi) \). Thus \( \hat{\varphi}(\xi) \) is an entire function in \( \mathbb{C}^n \).

\( \Rightarrow \) Since \( \varphi \) is in \( F_M \) we can use integration by parts in the following integration.

\[
\xi^\alpha \hat{\varphi}(\xi) = \int_{\mathbb{R}^n} D^\alpha (e^{-i\langle x, \xi \rangle}) \varphi(x) \, dx
\]

\[
= (-1)^{\left\lfloor \alpha \right\rfloor} \int_{\mathbb{R}^n} e^{-i\langle x, \xi \rangle} D^\alpha \varphi(x) \, dx.
\]

Therefore,

\[
|\xi|^{\left\lfloor \alpha \right\rfloor} |\varphi(\xi)| < \int_{\mathbb{R}^n} |D^\alpha \varphi(x)| e^{\langle \eta, x \rangle} \, dx
\]

\[
\leq C_K \int_{\mathbb{R}^n} e^{-M((k+1)x) + \langle \eta, x \rangle} \, dx
\]

\[
\leq C_k \sup_{x \in \mathbb{R}^n} (e^{-M(kx) + \langle \eta, x \rangle}) \int_{\mathbb{R}^n} e^{-M(x)} \, dx
\]

\[
\leq C_k e^{\alpha(1/k) \eta}
\]

Hence, for any given \( N \) and \( k \) there exists a constant \( C \) such that

\[
|\varphi(\xi)| \leq C (1 + |\xi|)^{-N} e^{\alpha(1/k) \eta}
\]

for any \( \xi \) in \( \mathbb{C}^n \).

\( \Leftarrow \) If \( F(\xi) \) is a given entire function which satisfies the growth condition in the hypothesis of (a) we can define the inverse Fourier transform \( \varphi(x) \) of \( F(\xi) \) by

\[
\varphi(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} F(\xi) e^{i\langle x, \xi \rangle} \, d\xi.
\]

The entire function \( F(\xi + i\eta) \) vanishes faster than any power of \( \frac{1}{|\xi|^k} \), as \( |\xi| \to \infty \), uniformly in any strip \( |\eta| \leq |\eta_0| \), and in this strip the absolute value of the function \( e^{i\langle x, \xi \rangle + i\eta} \) remains bounded. Therefore, using Cauchy’s theorem in the expression for the Fourier inverse transform
\[ \varphi(x) = \frac{1}{(2\pi)^{n}} \int_{\mathbb{R}^{n}} F(\xi) e^{i \langle x, \xi \rangle} d\xi, \]

one can replace the path of integration by any horizontal line for each coordinate, so that

\[ \varphi(x) = \frac{1}{(2\pi)^{n}} \int_{\mathbb{R}^{n}} F(\xi + i\eta) e^{i \langle x, \xi + i\eta \rangle} d\xi. \]

Differentiating this with respect to \( x \), we obtain

\[ D^\alpha \varphi(x) = \frac{1}{(2\pi)^{n}} \int_{\mathbb{R}^{n}} (i\xi)^{\alpha} F(\xi) e^{i \langle x, \xi \rangle} d\xi \]

where \( \xi = \xi + i\eta \in \mathbb{C}^{n} \) and the differentiation under the integral sign is legitimate due to the absolute convergence of the resulting integral. Therefore, we have

\[ |D^\alpha \varphi(x)| \leq \frac{1}{(2\pi)^{n}} \int_{\mathbb{R}^{n}} |\xi|^{|\alpha|} |F(\xi)| e^{-|\eta|} d\xi \]
\[ \leq \frac{C}{(2\pi)^{n}} \int_{\mathbb{R}^{n}} |\xi|^{|\alpha|} (1 + |\xi|)^{-N} e^{O(\eta) - |\eta|} d\xi \]
\[ \leq C e^{O(\eta) - |\eta|} \]

for some constant \( C \), provided that \( N > |\alpha| + n + 2 \).

Until now \( \eta \) has been an arbitrary number. Let us choose the sign of \( \eta \) in such a manner that the equality \( \langle x, \eta \rangle = \sum_{i=1}^{n} |x_i| |\eta_i| \) is satisfied. Using Young's inequality (Lemma 1.1), we obtain

\[ e^{O(\eta) - |\eta|} = e^{O(\eta)} - \sum_{i=1}^{n} |x_i| |\eta_i| = e^{-M^{X}} \]

and by choosing \( \varepsilon \) so small that \( \frac{1}{\varepsilon} > k \), we get

\[ \sup_{x \in \mathbb{R}^{n}} |e^{M(kx)} |D^\alpha \varphi(x)|| \leq \sup_{x \in \mathbb{R}^{n}} |e^{M(kx)} - M^{X}(x)| < \infty. \]

Thus \( \varphi(x) \in K_{M} \) and \( \hat{\varphi}(\xi) = F(\xi) \).

(b) \( \Rightarrow \) Let \( S \) be a given distribution in \( \mathcal{O}'(K_{M}', K_{M}') \). By Theorem 1.3.2, for every \( k = 1, 2, \ldots \), there exists a positive integer \( m \) such that

\[ S = \sum_{|\alpha| \leq m} \lambda f_{\alpha} \]

where \( f_{\alpha} \) are continuations in \( \mathbb{R}^{n} \) whose products with \( e^{M(kx)} \) are bounded. Because of the growth condition of \( f_{\alpha} \), the function

\[ F_{\alpha}(\xi) = \int_{\mathbb{R}^{n}} e^{-i \langle x, \xi \rangle} f_{\alpha}(x) dx \]
is an entire function, for each $\alpha$. Consequently

$$\hat{S}(\xi) = \sum_{|\alpha| \leq m} \xi^\alpha F_\alpha(\xi)$$

is an entire function. It suffices now to show that, for every $\varepsilon>0$, there is a constant $C$ such that

$$|F_\alpha(\xi)| \leq C e^{Q(\varepsilon \eta)}$$

for all $\xi$ in $\mathbb{C}^n$. Using the fact that $e^{M(kx)} f_\alpha(x)$ are bounded and (1.1), we obtain

$$|F_\alpha(\xi)| = |\int_{\mathbb{R}^n} e^{-i\langle x, \xi \rangle} f_\alpha(x) \, dx|$$

$$\leq C \int_{\mathbb{R}^n} e^{\langle x, \eta \rangle - M(kx)} \, dx$$

$$\leq C \int_{\mathbb{R}^n} e^{\langle x, \eta \rangle - M((k-\frac{1}{k})x) - M(\frac{1}{k}x)} \, dx$$

$$\leq C \sup_{x \in \mathbb{R}^n} \{ e^{\langle x, \eta \rangle - M((k-\frac{1}{k})x)} \int_{\mathbb{R}^n} e^{-M(\frac{1}{k}x)} \, dx \}$$

$$\leq C \sup_{x \in \mathbb{R}^n} \{ e^{\langle x, \eta \rangle - M((k-\frac{1}{k})x)} \}$$

$$\leq C e^{Q\left(\frac{k}{k^2 - 1}\right)}$$

where $C$ is some constant and we applied Lemma 1.3. Now, for any given $\varepsilon>0$, we can choose $k$ so large that $\frac{k}{k^2 - 1} < \varepsilon$. In this case, for all $|\alpha| \leq m$, we obtain

$$|F_\alpha(\xi)| \leq C e^{Q(\varepsilon \eta)},$$

and so

$$|\hat{S}(\xi)| \leq C(1 + |\xi|)^m e^{Q(\varepsilon \eta)}$$

for all $\xi$ in $\mathbb{C}^n$.

($\Leftarrow$). By hypothesis, the analytic function $F(\xi)$ is of polynomial growth and so $F$ is in $\mathcal{O}'$. If we denote $S$ the inverse Fourier transform of $F$, then $S$ is in $\mathcal{O}' \subset K'_M$.

We now show that $S$ is in $O'_\varepsilon(K'_M, K'_M)$. In order of Theorem 3.2, it is sufficient to show that $S \ast \varphi$ is in $K'_M$, whenever $\varphi$ is in $K'_M$. By part (a) of this theorem it is equivalent to show that $\hat{S} \hat{\varphi}$ is an entire function and, for every constants $N$ and $\varepsilon>0$, there exists a constant $C>0$ such that
\[ |\hat{S} \ast \varphi(\zeta)| \leq C |\zeta|^{-N} e^{O(\eta)} \]

for all \( \zeta \in \mathbb{R}^n \setminus \{0\} \).

We have the following two estimations.

For any given \( \epsilon_1 > 0 \), there exist constants \( C \) and \( k_1 \) such that

\[ |\hat{S}(\zeta)| \leq C (1 + |\zeta|)^{k_1} e^{O(\eta)} \]

for all \( \zeta \in \mathbb{R}^n \) by our hypothesis of \( F(\zeta) \) and, since \( \varphi \in K_M \), for any given \( \epsilon_1 > 0 \) and \( k_2 \) there exists a constant \( C > 0 \) such that

\[ |\hat{\varphi}(\zeta)| \leq C (1 + |\zeta|)^{-k_2} e^{O(\eta)} \]

for all \( \zeta \in \mathbb{R}^n \) by (a). Therefore, for any given \( \epsilon \) and \( N \), if we choose \( \epsilon_1 \) and \( \epsilon_2 \) so that \( \epsilon_1 + \epsilon_2 \leq \epsilon \) and \( k_2 \geq k_1 + N \), we obtain

\[ |\hat{S} \ast \varphi(\zeta)| = |\hat{S}(\zeta)| |\hat{\varphi}(\zeta)| \leq C (1 + |\zeta|)^{k_1} e^{O(\eta)} (1 + |\zeta|)^{-k_2} e^{O(\eta)} \]
\[ \leq C (1 + |\zeta|)^{k_1 - k_2} e^{O(\eta)} (1 + |\zeta|)^{k_1} e^{-\Omega(\eta)} \]
\[ \leq C (1 + |\zeta|)^{-N} e^{O(\eta)} \]

for some constant \( C \). This implies that \( S \ast \varphi \in K_M \) for every \( \varphi \in K_M \), that is, \( S \) is in \( O'(K'_M, K'_M) \).

Let \( K_M \) be the space of Fourier transforms of functions in \( K_M \). We define in \( K_M \) a locally convex topology by means of the semi-norms

\[ w_k(\varphi) = \sup_{\zeta \in \mathbb{R}^n} |\zeta|^k e^{-\Omega(\eta)} |\hat{\varphi}(\zeta)|, \quad k = 1, 2, \ldots \]

for all \( \varphi \) in \( K_M \).

**Corollary 4.2.** The Fourier transformation is a topological isomorphism of \( K_M \) onto \( K_M \).

**Proof.** By Theorem 4.1 and since the Fourier inversion formula is valid for functions in \( K_M \), the Fourier transformation is an isomorphism of \( K_M \) onto \( K'_M \).

In view of the open mapping theorem it therefore suffices to show that the mapping \( \varphi \rightarrow \hat{\varphi} \) of \( K_M \) onto \( K'_M \) is continuous. We observe that if \( \zeta \) is any given integer \( \geq 0 \) and \( \alpha \) is any given multi-index with \( |\alpha| \leq k \),

\[ |\hat{\varphi}(\zeta)| = |\int_{\mathbb{R}^n} e^{-i(x \cdot \xi)} D^\alpha \varphi(x) dx| \]
Structure Theorem and Fourier Transform for distributions with restricted growth

\[ \omega_k(\varphi) \leq C \nu_{k+1}(\varphi), \text{ by Lemma 1.1.3.} \]

Therefore

\[ \omega_k(\varphi) \leq C \nu_{k+1}(\varphi) \]

for some constant \( C \) which does not depend on \( \varphi \in K_M \), this gives the desired continuity.

Let \( H' \) be the space of continuous linear functional on \( H_M \), which equipped with the topology of uniform convergence on all bounded sets in \( H_M \).

Each distribution \( T \) in \( K'_M \) has a Fourier transform \( \hat{T} \) in \( K'_M \) defined by Parseval's formula

\[ \langle \hat{T}, \varphi \rangle = (2\pi)^n \langle T, \varphi' \rangle \]

for all \( \varphi \in K_M \). Moreover, by Corollary 1.4.2, we have

**COROLLARY 4.3.** The Fourier transformation is a topological isomorphism of \( K'_M \) onto \( H'_M \).

If \( S \in \mathcal{O}'(K'_M, K'_M) \), then by Theorem 4.1 the mapping \( \mathcal{F} \rightarrow \mathcal{F} \mathcal{F} \) is a continuous linear mapping of \( H_M \) into \( H_M \). Therefore, if \( T \) is the Fourier transform of a distribution \( T \in K'_M \), we can define the product \( \hat{T} \) by

\[ \langle \hat{T}, \mathcal{F} \rangle = \langle T, \mathcal{F} \mathcal{F} \rangle \]

for every \( \mathcal{F} \in H_M \). Moreover we can easily prove that

\[ (S \star T) = \hat{S} \hat{T} \].

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