GENERALIZED ASCENDING CHAINS AND FORMAL POWER SERIES RINGS

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1. Introduction

Generalized ascending chains of ideals arise naturally in polynomial rings in several variables. By using such ideals we were able to prove the Hilbert basis theorem for a polynomial ring in several variables without using the usual method of extending variables. In this paper we will prove a similar result for a formal power series ring in several variables. This will be done by using a modification of the argument used to prove the theorem for a polynomial ring in several variables.

Let \( P \) be the set of non-negative integers and \( P^n \) the product of \( P \) \( n \)-times. If \( \alpha = (i_1, \ldots, i_n) \) and \( \beta = (j_1, \ldots, j_n) \in P^n \), then we will say that

(i) \( \alpha = \beta \) if \( i_k = j_k \) for each \( k \in \{1, 2, \ldots, n\} \); and

(ii) \( \alpha < \beta \) if \( i_k < j_k \) for some \( k \in \{1, 2, \ldots, n\} \) and

\[ i_t \leq j_t \text{ for all } t \neq k. \]

If either \( \alpha = \beta \) or \( \alpha < \beta \), then we will say that \( \alpha \leq \beta \). We define the sum of \( \alpha \) and \( \beta \) to be \( \alpha + \beta = (i_1 + j_1, \ldots, i_n + j_n) \).

1.1. DEFINITION. A collection of ideals \( \{A_\alpha | \alpha \in P^n\} \) in a ring \( R \) will be called a generalized ascending chain of dimension \( n \) if whenever \( A_\alpha \) and \( A_\beta \) are two ideals in the collection with \( \alpha \leq \beta \), then \( A_\alpha \subseteq A_\beta \). The generalized ascending chain of ideals \( \{A_\alpha | \alpha \in P^n\} \) is called finite if there is a \( \Delta = (N, N, \ldots, N) \in P^n \) such that for each \( \alpha = (i_1, i_2, \ldots, i_n) \in P^n \) with \( \max_i i_k \geq N \) we have \( A_\alpha = A_\beta \) for some \( \beta \leq \Delta \). If we consider the elements of \( P^n \) to be lattice points, then to say that \( \{A_\alpha | \alpha \in P^n\} \) is finite means that there is an \( n \)-dimensional cube \( C_N \) of length \( N \) such that for any \( \alpha \in P^n \) and \( \alpha \not\in C_N \) there is a \( \beta \in C_N \) with \( A_\alpha = A_\beta \).

It was proved in [1] that if \( R \) is a Noetherian ring, then every generalized ascending chain of ideals in \( R \) is finite. If \( R_1, R_2, \ldots, R_n \) are rings, then generalized ascending chains arise naturally in the direct sum \( R = R_1 \oplus R_2 \oplus \cdots \oplus R_n \). It is
easy to show that every generalized ascending chain in $R$ is finite if and only if every generalized ascending chain in each $R_i$ is finite.

2. Power series rings in several variables.

Let $R$ be a commutative ring with an identity and $R[[x_1, x_2, \ldots, x_n]]$ the formal power series ring in the indeterminates $x_1, x_2, \ldots, x_n$. An element in $R[[x_1, x_2, \ldots, x_n]]$ is of the form $f=\Sigma a_{i_1 \cdots i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ with each $i_k$ being unrestricted. The coefficients of $f$ are also unrestricted. Let $\alpha=(i_1, i_2, \ldots, i_n)$, $|\alpha|=i_1+i_2+\cdots+i_n$ and write $a_{\alpha}=a_{i_1 \cdots i_n}$ and $X^\alpha=x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$. Then the polynomial $f$ may be written $f=\Sigma a_{\alpha} X^\alpha$ with $|\alpha|\to \infty$. The degree of a non-zero term $a_{\alpha} X^\alpha$ of $f$ is $|\alpha|$. Now for each $\alpha$ with $|\alpha|=m$, $f$ may contain more than one term of degree $m$. If $f_m$ is the sum of all terms of $f$ of degree $m$, then clearly $f_m$ is a homogeneous polynomial. Consequently, we can write $f=\Sigma_{0}^{\infty} f_m$, where $f_m$ is a homogeneous polynomial of degree $m$. Thus each non-zero polynomial $f$ contains a homogeneous polynomial of lowest degree. For each polynomial $f_m$, we consider the lexicographic ordering for the terms of $f_m$, i.e., if $a_{\alpha} X^\alpha$ and $a_{\tau} X^\tau$ are terms of $f_m$ with $\alpha=(i_1, i_2, \ldots, i_n)$ and $\tau=(t_1, t_2, \ldots, t_n)$ such that $i_1=t_1, i_2=t_2, \ldots, i_s=t_s$, but $i_{s+1}>t_{s+1}$ ($s\geq0$), then we say that $a_{\alpha} X^\alpha$ is higher than $a_{\tau} X^\tau$ or $a_{\alpha} X^\alpha$ is lower than $a_{\tau} X^\tau$. It is clear that $f_m$ can have only one lowest term. Consequently, the polynomial $f$ can have only one lowest term of lowest degree. We will call such a term the lowest term of $f$. Now if $H$ is an ideal in $R[[x_1, x_2, \ldots, x_n]]$, let

$$H_{\alpha}=\{b \in R|b=0 \text{ or } bX^\alpha \text{ is the lowest term of some } f \in H\}.$$

2.1. LEMMA. If $H$ is an ideal in $R[[x_1, x_2, \ldots, x_n]]$, then $[H_{\alpha}|\alpha \in P^n]$ is a generalized ascending chain of ideals in $R$.

PROOF. If $a, b \in H_{\alpha}$ and $r \in R$, then $a-b$ and $ra$ are elements in $H_{\alpha}$ as one sees by taking the difference of the corresponding polynomials and $r$ times the corresponding polynomial. Consequently, each $H_{\alpha}$ is an ideal. Now let $0 \neq b \in H_{\alpha}$ and $\beta \in P^n$ such that $\alpha<\beta$. If $\alpha=(i_1, i_2, \ldots, i_n)$ and $\beta=(j_1, j_2, \ldots, j_n)$ then $j_k=i_k+t_k$ for some $t_k\geq0$. Let $\tau=(t_1, t_2, \ldots, t_n)$ and $f$ be a polynomial in $H$ with $bX^\alpha$ as lowest term. Then $X^\tau bX^\alpha=bX^{\alpha+\tau}=bX^\beta$ is the lowest term.
of the polynomial $xf$. Consequently, $b \in H_\beta$ and it follows that $H_\alpha \subseteq H_\beta$. Therefore $\{H_\alpha | \alpha \in P^m\}$ is a generalized ascending chain of ideals in $R$. The ideals $\{H_\alpha | \alpha \in P^m\}$ will be called the lowest coefficient ideals of $H$.

2.2. THEOREM. If $R$ is a Noetherian ring, then $R[[x_1, x_2, \ldots, x_n]]$ is also Noetherian.

PROOF. Let $H$ be an ideal in $R[[x_1, x_2, \ldots, x_n]]$ and $\{H_\alpha | \alpha \in P^m\}$ the corresponding lowest coefficient ideals. It follows from Lemma 2.1 that this collection of ideals is a generalized ascending chain of ideals in $R$ and since $R$ is Noetherian this collection is finite, i.e., there exists $\Delta = (N, N, \ldots, N)$ such that for each $\alpha = (i_1, i_2, \ldots, i_n) \in P^m$ with $\max k i_k \geq N$, $H_\alpha = H_\beta$ for some $\beta \leq \Delta$.

Since $R$ is Noetherian, each $H_\alpha$ is finitely generated, say by elements $b_{\alpha 1}, b_{\alpha 2}, \ldots, b_{\alpha m}$. By the Axiom of Choice, for each $\alpha \leq \Delta$ and $k \in \{1, 2, \ldots, m\}$ we can choose a polynomial $f_{\alpha k}$ in $H$ with $b_{\alpha k}$ as the coefficient of its lowest term.

The proof of the theorem will be completed by showing that the finite set \{f_{\alpha k} : 1 \leq k \leq m, \alpha \leq \Delta\} generates $H$. To this end, consider a typical polynomial $f = \sum a_k X^k$ in $H$ with lowest term $a_1 X^r$. Let $|\tau| = r$. Then the least degree of $f$ is $r$. If $\tau = (t_1, t_2, \ldots, t_n)$, then $t_j > N$ for some $j$ or $t_j \leq N$ for each $j$. If $t_j > N$ for some $j$, then there exists $\alpha \leq \Delta$ such that $\tau > \alpha$ and $H_\tau = H_\alpha$. Hence the lowest coefficients of the polynomials

$$X^{i-\alpha} f_{\alpha 1}, X^{i-\alpha} f_{\alpha 2}, \ldots, X^{i-\alpha} f_{\alpha m},$$

generate $H_\tau$. Thus there are elements $c_{\tau 1}, c_{\tau 2}, \ldots, c_{\tau m}$ in $R$ such that the lowest term of $f_1 = f - \sum c_{\tau k} f_{\alpha k}$ is higher than that of $f$ or the least degree of $f_1$ is higher than $r$, and $f_1$ lies in $H$. If $f$ and $f_1$ have the same least degree $r$, then we can repeat the process with $f_1$. Since there are only a finite number of terms of $f$ of degree $r$ that are higher than a given one, a finite number of applications of this process will yield a polynomial $g = f - \sum c_{\tau k} f_{\alpha k}$, the sum taken over all $\tau$ with $|\tau| = r$ and corresponding $\alpha$, such that the least degree of $g$ is greater than $r$ and $g$ lies in $H$. On the other hand, if $t_j \leq N$ for all $j$, then $\tau \leq \eta$ and a process similar to the above will yield a polynomial $g' = f - \sum c_{\eta k} f_{\alpha k} \in H$ such that the least degree of $g'$ is greater than $r$. Consequently, by induction on the least degree of $f$ we can find a polynomial $h \in H$ generated by $\{f_{\alpha k}, \alpha \leq \Delta\}$ such that $f - h = 0$. Therefore $H$ is generated by $\{f_{\alpha k}, \alpha \leq \Delta\}$.
and it follows that $R[[x_1, x_2, \ldots, x_p]]$ is Noetherian.

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REFERENCES
