THE EXISTENCE AND UNIQUENESS OF THE SOLUTION OF A
BOUNDARY VALUE PROBLEM FOR A VECTOR DIFFERENTIAL
EQUATION OF nTH ORDER IN BANACH SPACE

By M. Sobeh and G. Hamad

1. Introduction:

The problem which we study here, is firstly suggested by Niclibork [4]. The problem deals with a projectile with a given initial velocity hits a target. It was formulated as an initial boundary value problem with a system of two differential equations of second order. In 1966 Perov and Makhmoudov [6] generalized the problem for a system of n-equations using vector notations. Many other different generalizations for this problem are suggested and studied in [1, 2, 7, 8, 9].

In this work, we mainly adjoin the works of Bagirian [1] and Sobeh [7] getting a more generalized formulation to the same problem.

2. Formulation of the problem:

Find the solution \((x(t), t_*)\) for the following problem:

\[
x^{(n)} = f(t, x, x^{(1)}, \ldots, x^{(n-1)}); t \in [0, T],
\]

\[
x^{(i)}(0) = 0; i \neq i_0; 0 < i_0 \leq n - 1 \quad (i = 0, n - 1; n \geq 2),
\]

\[
\|x^{(i)}(0)\| = v > 0,
\]

and

\[
\sum_{j=0}^{n-1} [\alpha_j x^{(j)}(t_*)] + \int_0^T A_j(s, t_*) x^{(j)}(s) ds +
\]

\[
+ \int_0^T B_j(s, t_*) x^{(j)}(s) ds = x^*; t_* \in (0, T).
\]

where \(f(t, y_0, y_1, \ldots, y_{n-1})\) and its arguments \(y_0, y_1, \ldots, y_{n-1}\) are vector functions in Banach space \(E\), together with the given constant vector \(x^* \neq 0\). \(v, T\) are given positive numbers. Also \(\alpha_j\) \((j = 0, 1, \ldots, n - 1)\) are \(n \times n\) constant matrices and \(A_j(t, s), B_j(t, s)\) \((j = 0, 1, \ldots, n - 1)\) are \(n \times n\) matrix functions satisfying \(B_j(t, s)_{t=0} = 0.\)
Here if \( n=2 \) the problem in [7] is obtained, while if \( A_j(t, s)=0 \) and \( B_j(t, s)=0 \) \((j=0, 1, \ldots, n-1)\) we can get [1]. As usual, we denote by \( C^n([0, T]; E) \) the set of all \( n \)-times differentiable and continuous vector functions in \([0, T]\) with values in \( E \). And hence, we can define the first argument of the solution of problem (1)—(4) as the vector function \( x(t) \) which belongs to \( C^n \) and satisfies conditions (2)—(4).

3. Preliminaries:

(i) Suppose that \( f(t, y_0, y_1, \ldots, y_{n-1}) \) is defined and continuous in 
\[ R : ([0, T], \|y_j\| \leq a_i^* (i=0, 1, \ldots, n-1)) \]
and satisfies the following two conditions:
\[
\max_{0 \leq t \leq T} \|f(t, y_0, y_1, \ldots, y_{n-1})\| \leq M, \tag{5}
\]
and
\[
\|f(t, x_0, x_1, \ldots, x_{n-1})-f(t, y_0, \ldots, y_{n-1})\| \leq \sum_{j=0}^{n-1} L_j \|x_j-y_j\| \tag{6}
\]
where \( M, L_j, j=0, 1, \ldots, n-1 \) are constants.

(ii) Consider the following functions:
\[
\phi_q(t) = \int_0^t \frac{(t-s)^{n-q-1}}{(n-q-1)!} f(s) ds, \tag{7}
\]
\[
y(t) = \sum_{q=0}^{n-1} \left[ \frac{1}{q!} \int_0^t A_q(s, t) \int_0^s \frac{(s-s_1)^{n-q-1}}{(n-q-1)!} f(s_1) ds_1 ds + \int_0^T B_q(s, t) \int_0^s \frac{(s-s_1)^{n-q-1}}{(n-q-1)!} f(s_1) ds_1 ds \right], 0 \leq s \leq t \leq T, \tag{8}
\]
\[
P(t) = \sum_{q=0}^{i_q} \left[ \frac{\alpha_q^{i_q-q}}{q!} t^{i_q-1} + \frac{1}{t} \int_0^t \frac{A_q(i_q-q)}{q!} s^{i_q} ds + \frac{1}{t} \int_0^T \frac{B_q(i_q-q)}{q!} s^{i_q} ds \right],
\]
\[
S(t) = P^{-1}(t) ; t \neq 0.
\]
\[
f(s) = f(s, x(s), x^{(1)}(s), \ldots, x^{(n-1)}(s)),
\]
which can be easily shown that they are continuous in their intervals of definitions. Also, suppose that they satisfy the conditions:
\[
\|\phi_q(t_1)-\phi_q(t_2)\| \leq K_1 |t_1-t_2|,
\]
\[
\|y(t_1)-y(t_2)\| \leq K_2 |t_1-t_2|, \quad t_1, t_2 \in [0, T].
\]

\( \Box \)
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\[ \|S(t_1) - S(t_2)\| \leq K_3 |t_1 - t_2|, \quad t_1, t_2 \in (0, T) \]

4. Theorem of existence and uniqueness:
Using the following definitions:
\[ \delta = \max \|x^*\|, \quad \Delta_q = \max \|\alpha_q\|, \]
\[ a_q = \max_{0 \leq s, t \leq T} \|A_q(s, t)\|, \quad \bar{a}_q = \max_{0 \leq s, t \leq T} \|\frac{\partial A_q(s, t)}{\partial t}\|, \]
\[ b_q = \max_{0 \leq s, t \leq T} \|B_q(s, t)\|, \quad \bar{b}_q = \max_{0 \leq s, t \leq T} \|\frac{\partial B_q(s, t)}{\partial t}\|, \]
\[ C_q = a_q + b_q, \quad \bar{C}_q = \bar{a}_q + \bar{b}_q. \]

\[ \Phi_q = \max_{0 \leq t \leq T} \|\phi_q(t)\|, \quad Y = \max_{0 \leq t \leq T} \|y(t)\|, \quad K = \max_{0 < t \leq T} \|S(t)\| \quad q = 0, 1, \ldots, n-1, \]

we can introduce the following existence and uniqueness theorem.

THEOREM: If conditions (5) and (6) are satisfied, together with the following conditions:

\[ \mu = K\left(\delta + \sum_{q=0}^{n-1} \Delta_q \Phi_q + Y\right) \leq vT, \]

\[ \rho_1 = \frac{1}{v} \left[ K_3 (\delta + \sum_{q=0}^{n-1} \Delta_q \Phi_q + Y) + K (K_2 + K_1 \sum_{q=0}^{n-1} \Delta_q) \right] < 1, \]

\[ \frac{T^{i+1}}{(i+1)!} v + \frac{T^n}{(n-i)!} M \leq a_i^*; \quad i = 0, \ldots, i_0, \]

\[ \frac{T^{n-i}}{(n-i)!} M \leq a_i^*; \quad i = i_0 + 1, \ldots, n-1 \]

and

\[ \sum_{i=0}^{i_0} L_i \left[ \frac{T^{i+1}}{(i+1)!} \left[ \frac{2vK}{\rho_2 Q} \sum_{q=0}^{n-1} \frac{T^{n-q}}{(n-q)!} \left( \Delta_q + \frac{C_q T}{(n-q+1)} \right) \right] + \frac{T^{n-i}}{(n-i)!} \right] \leq \sum_{i=i_0+1}^{n-1} \frac{T^{n-i}}{(n-i)!} L_i < 1 \]

where

\[ \rho_2 = \frac{h}{v + K (K_2 + K_1 \sum_{q=0}^{n-1} \Delta_q)} \quad h = \min_{0 < t \leq T} \|S(t)x^*\|, \]

\[ Q = v - K \left( \delta + \sum_{q=0}^{n-1} \Delta_q \Phi_q + Y \right) - K (K_2 + K_1 \sum_{q=0}^{n-1} \Delta_q). \]
Then problem (1)–(4) has a unique solution \((x(t), t_*)\).

5. Some lemmas:

To prove the previous existence and uniqueness theorem we need the following four lemmas.

**Lemma 1:** If functions \(\phi_q(t)\) \((q=0, 1, \ldots, n-1)\), \(y(t)\) and \(S(t)\) satisfy conditions (10) and if conditions (11), (12) are satisfied, then the equation

\[
t = \frac{1}{v} \|S(t)\left(x^* - \sum_{q=0}^{n-1} \alpha_q \phi_q(t) - y(t)\right)\|
\]

has a unique solution \(t_* \in (0, T]\).

**Proof:** Let the operator \(F(t)\) denotes the right hand side of equation (15). Therefore

\[0 < F(t) \leq \frac{1}{v} \mu\]

and

\[|F(t_1) - F(t_2)| \leq \rho_1 |t_1 - t_2|\]

Consequently according to conditions (11), (12) the operator \(F(t)\) maps the interval \((0, T]\) into itself and satisfies the condition of contraction, and hence, equation (15) has a unique solution \(t_* \in (0, T]\).

**Note:** If \(t_*\) is the minimal value of \(t\) in equation (15) and if \(\phi_q(0) = 0\) \((q=0, 1, \ldots, n-1)\), \(y(0) = 0\), then we can estimate that

\[t_* \geq \frac{h}{v + K(K_2 + K_1 \sum_{q=0}^{n-1} \Delta_q)} = \rho_2 > 0\]

**Lemma 2:** Let the vector functions \((\phi_q(t), y(t))\) and \((\psi_q(t), Z(t))\); \(q=0, 1, \ldots, n-1\) correspond respectively to the roots \(t_1\) and \(t_2\) of the equations:

\[
t = \frac{1}{v} \|S(t)\left(x^* - \sum_{q=0}^{n-1} \alpha_q \phi_q(t) - y(t)\right)\|
\]

\[
t = \frac{1}{v} \|S(t)\left(x^* - \sum_{q=0}^{n-1} \alpha_q \psi_q(t) - Z(t)\right)\|
\]

and satisfy conditions (10)–(12). Also if \(\eta_1, \eta_2\) are two vector functions of the form:

\[\eta_1 = \frac{S(t_1)}{t_1} \left\{ x^* - \sum_{q=0}^{n-1} \alpha_q \phi_q(t_1) - y(t_1) \right\}\]

\[\eta_2 = \frac{S(t_2)}{t_2} \left\{ x^* - \sum_{q=0}^{n-1} \alpha_q \psi_q(t_2) - Z(t_2) \right\}\]
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\[ \eta_2 = \frac{S(t_2)}{t_2} \left\{ x^* - \sum_{q=0}^{n-1} \alpha_q \Psi_q(t_2) - Z(t_2) \right\}, \] (17)

then we can estimate that:

\[ |t_1 - t_2| \leq \frac{KW}{\omega l} \quad \text{and} \quad \|n_1 - n_2\| \leq \frac{(2\nu K)W}{\rho_2 Q} \] (18)

where \( W = \sum_{q=0}^{n-1} \Delta_q \max_{0 \leq t \leq T} \|\phi_q(t) - \Psi_q(t)\| + \max_{0 \leq t \leq T} \|y(t) - Z(t)\| \) (19)

and \( K, Q, \rho_2 \) are defined as before.

**PROOF:** Since \( t_1, t_2 \) are the roots of equations (16), then

\[ |t_1 - t_2| \leq \frac{1}{n} \|S(t_1) [x^* - \sum_{q=0}^{n-1} \alpha_q \phi_q(t_1) - y(t_1)] - S(t_2) [x^* - \sum_{q=0}^{n-1} \alpha_q \Psi_q(t_2) - Z(t_2)] \|, \]

which from (10) gives

\[ v|t_1 - t_2| \leq [k_2(\beta + \sum_{q=0}^{n-1} \Delta_q \phi_q + \gamma) + k(k_2 + k_1 \sum_{q=0}^{n-1} \Delta_q)] |t_1 - t_2| + k(\sum_{q=0}^{n-1} \Delta_q) \times \max_{0 \leq t \leq T} \|\phi_q(t) - \Psi_q(t)\| + \max_{0 \leq t \leq T} \|y(t) - Z(t)\|. \]

Therefore, using (19) the last inequality leads to the first inequality of (18).

Also form (17), using the well known inequality

\[ \|A - B\| \leq \frac{2\|A - B\|}{\max \{\|A\|, \|B\|\}} \]

we can obtain

\[ \|n_1 - n_2\| \leq \frac{2}{\gamma} \|S(t_2) - S(t_2)\| x^* - \sum_{q=0}^{n-1} \alpha_q S(t_1) \phi_q(t_1) - S(t_2) \Psi_q(t_2) - S(t_1) y(t_1) - S(t_2) Z(t_2) \| \]

where

\[ \gamma = \max \{\|S(t_1) [x^* - \sum_{q=0}^{n-1} \alpha_q \phi_q(t_1) - y(t_1)]\|, \]

\[ \|S(t_2) [x^* - \sum_{q=0}^{n-1} \alpha_q \Psi_q(t_2) - Z(t_2)]\|. \]

Using (10) and (19) we can have

\[ \|n_1 - n_2\| \leq \frac{2(v - Q)|t_1 - t_2| + KW}{\rho_2} \]
and hence, substituting from the first inequality of (18), the second inequality of (18) will be hold.

**LEMMA 3.** If the vector functions \( \phi_q(t) (q=0, 1, \ldots, n-1) \) and \( y(t) \) are defined as in (7), (8), then the following is true:

\[
\begin{align*}
\max_{0 \leq t \leq T} \| \phi_q(t) \| & \leq M \frac{T^{n-q}}{(n-q)!}, \\
\max_{0 \leq t \leq T} \| \phi'_q(t) \| & \leq M \frac{T^{n-q-1}}{(n-q-1)!}, \\
\max_{0 \leq t \leq T} \| y(t) \| & \leq M \sum_{q=0}^{n-1} \frac{T^{n-q+1}}{(n-q+1)!} C_q, \\
\max_{0 \leq t \leq T} \| y'(t) \| & \leq M \sum_{q=0}^{n-1} \frac{T^{n-q}}{(n-q)!} \left( a_q + \frac{C_q T}{n-q+1} \right),
\end{align*}
\]

where \( M, a_q, C_q, \bar{C}_q \) are defined as before.

**LEMMA 4:** Let \( P(t) \) be the non-singular matrix function which is defined in (9), then for its inverse matrix \( S(t) \) we have

\[
\begin{align*}
\max_{0 < t < T} \| S(t) \| & \leq \frac{n^2}{N} (n-1)^2 \left[ \sum_{q=0}^{n-1} \frac{T^{q-1}}{q!} \left( \Delta_{i_q-a_q} + \frac{C_{i_q-a_q} T}{q+1} \right) \right]^{(n-1)}, \\
\max_{0 < t < T} \| S'(t) \| & \leq \frac{n^4}{N^2} (n-1)^{n-2} \left[ \sum_{q=0}^{n-1} \frac{T^{q-1}}{q!} \left( \Delta_{i_q-a_q} + \frac{C_{i_q-a_q} T}{q+1} \right) \right]^{2(n-1)} \times \\
& \quad \times \sum_{q=0}^{n-1} \frac{T^{q-1}}{(q+1)!} \left[ \Delta_{i_q-a_q} + a_{i_q-a_q} \frac{T}{q} + (C_{i_q-a_q} + \bar{C}_{i_q-a_q}) \frac{T}{q(q+1)} \right],
\end{align*}
\]

where \( N = \min_{0 < t < T} | \det P(t) |. \)

**PROOF:** The inequality (24) is easily followed by using Adamar's inequality [3]. Also, since

\[
S'(t) = - S(t) P'(t) S(t),
\]

then the inequality (25) can be easily obtained.

**6. Proof of the theorem of Existence and uniqueness:**

Let \( U \subseteq C^n \), where

\[
U : \{ [0, T], \| x^{(i)} (t) \| \leq a_i^* (i=0, 1, \ldots, n-1) \}.
\]

Suppose that \( x(t) \in U \)

It is easily to see that problem (1)–(4) is equivalent to the system
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\[ x(t) = \eta_0 + \sum_{i=1}^{n} \eta_i \frac{(t-s)^{n-1}}{(n-1)!} f(s)ds, \]  
\[ \|\eta\| = \frac{1}{t_*^n} \|S(t_*)\| [x^{(n)} - \sum_{q=0}^{n-1} \alpha_q \phi_q(t_*) - y(t_*)] \],

where \( \phi_q(t_*) \) (\( q=0, 1, \ldots, n-1 \)), \( y(t_*) \), \( S(t_*) \) are defined as in (7), (8), (9) respectively.

According to conditions (11), (12) for every element \( x(t) \in U \) equation (26-2) has a unique solution \( t_*(x) \).

Now we define in \( U \) the operator \( D \) such that

\[ DX(t) = \frac{t^i \eta(t)}{t_0^i} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s)ds, \]

where \( \eta(t) = \frac{S(t^*_x(t))}{t^*_x(t)} [x^{(n)} - \sum_{q=0}^{n-1} \alpha_q \phi_q(t^*_x(t)) - y(t^*_x(t))] \].

Therefore,

\[ \|D^{(i)} x(t)\| \leq \frac{T^{i_0-i}}{(i_0-i)!} v + M \frac{T^{n-i}}{(n-i)!} : i=0, 1, \ldots, i_0 \]

and

\[ \|D^{(i)} x(t)\| \leq M \frac{T^{n-i}}{(n-i)!} : i=i_0+1, \ldots, n-1 \]

and hence from (13) it follows that \( D^{(i)} x(t) \in U \) (\( i=0, 1, \ldots, n-1 \)) i.e. the operator \( D \) maps the set \( U \) into itself.

Let \( x_1(t), x_2(t) \in U \), then

\[ \|D^{(i)} x_1(t) - D^{(i)} x_2(t)\| \leq \|\eta(x_1) - \eta(x_2)\| \frac{T^{i_0-i}}{(i_0-i)!} + \]

\[ + \frac{T^{n-i}}{(n-i)!} \sum_{j=0}^{n-1} L_j \|x_1^{(j)}(t) - x_2^{(j)}(t)\| : i=0, 1, \ldots, i_0 \]

and

\[ \|D^{(i)} x_1(t) - D^{(i)} x_2(t)\| \leq \frac{T^{n-i}}{(n-i)!} \sum_{j=0}^{n-1} L_j \|x_1^{(j)}(t) - x_2^{(j)}(t)\| : i=i_0+1, \ldots, n-1 \]

Using Lemmas (2) – (4) we can obtain

\[ \max_{0 \leq t \leq T} \|D^{(i)} x_1(t) - D^{(i)} x_2(t)\| \leq \left[ \frac{2vK}{\rho_2 Q} \sum_{q=0}^{n-1} \frac{T^{n-q}}{(n-q)!} \right] \Delta_q + \]
and
\[
\max_{0 \leq t \leq T} \| D^{(i)} x_1(t) - D^{(i)} x_2(t) \| \leq \frac{T^{n-i}}{(n-i)!} \sum_{j=0}^{n-1} \max_{0 \leq t \leq T} \| x_1^{(j)}(t) - x_2^{(j)}(t) \| ;
\]
\[
: i = i_0 + 1, \ldots, n - 1
\]

Introducing the generalized norm [5] by the equality
\[
|x| = \begin{vmatrix}
\max \| x \| \\
\max \| x^{(1)} \| \\
\vdots \\
\max \| x^{(n-1)} \|
\end{vmatrix},
\]

The inequalities in (27-1), (27-2) can be put in the form
\[
|Dx_1 - Dx_2| \leq S^*|x_1 - x_2|,
\]

where
\[
S^* = \begin{bmatrix}
u_0 L_0 & u_0 L_1 & \cdots & u_0 L_{n-1} \\
u_1 L_0 & u_1 L_1 & \cdots & u_1 L_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
u_{n-1} L_0 & u_{n-1} L_1 & \cdots & u_{n-1} L_{n-1}
\end{bmatrix},
\]
\[
u_i = \frac{T^{i_o-i}}{(i_o-t)!} \left[ 2 \nu_k \frac{T^{n-q}}{(n-q)!} \left( \Delta q + \frac{C_q T}{\Delta q + 1} \right) \right]
\]
\[
+ \frac{T^{n-i}}{(n-i)!} ; \quad i = 0, 1, \ldots, i_0
\]

and
\[
u_i = \frac{T^{n-i}}{(n-i)!} ; \quad i = i_0 + 1, \ldots, n - 1.
\]

It is easily to verify that $S^*$ is the a-matrix [5] if and only if (14) is satisfied. Consequently, according to the generalization of principle of contraction mapping [5], we can say that the operator $D$ has in $U$ a unique fixed point $x(t)$.

From above, it follows that problem (1)-(4) has a unique solution $(x(t), t^*)$. 
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This completes the proof of the theorem.

REMARK: Again problem (1)–(4) will be considered later to study the stability of its solution, whose existence and uniqueness have been proved here.

Dept. of Math, Faculty of Science, Al-Azhar University, Nasr City, Cairo, Egypt.

REFERENCES