Bull. Korean Math. Soc. **21** (1984), No. 1, 1~6

LINEAR MAPPINGS ON LINEAR 2-NORMED SPACES

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The notion of linear 2-normed spaces was introduced by S. Gahler ([8, 9, 10, 11]), and these spaces have been extensively studied by C. Diminnie, R. Ehret, S. Gahler, K. Iseki, A. White, Jr. and others.

Let X be a real linear space of dimension greater than 1 and v a real valued function on $X\times X$ satisfying the following axioms:

(1) $v(x, y)=0$ if and only if x and y are linearly dependent,
(2) $v(x, y)=v(y, x)$,
(3) $v(ax, y)=|a|v(x, y)$, where a is real,
(4) $v(x, y+z)\leq v(x, y)+v(x, z)$.

v is called a 2-norm on X and (X, v) is called a linear 2-normed space. Some of the basic properties of 2-norms are that they are nonnegative and $v(x, y+ax)=v(x, y)$ for every x, y in X and every real a.

Linear 2-normed spaces are special cases of a larger class called 2-metric spaces. A 2-metric space is a space X with a real-valued nonnegative function d defined on $X\times X\times X$ which satisfies the following conditions:

(1a) For each pair of points x, y in X with $x\neq y$, there exists a point z in X such that $d(x, y, z)\neq 0$,
(b) $d(x, y, z)=0$ whenever at least two of the points x, y, z are equal,
(2) $d(x, y, z)=d(x, z, y)=d(y, z, x)$,
(3) $d(x, y, z)\leq d(x, y, w)+d(x, w, z)+d(w, y, z)$.

d is called a 2-metric for the space X and (X, d) is called a 2-metric space. For more details on these spaces, see [8, 9, 10, 11]. If (X, v) is a linear 2-normed space, then the function $d(x, y, z)=v(x-z, y-z)$ defines a 2-metric on X([9]). Every 2-normed space will be considered to be a 2-metric space with the 2-metric defined in this sense.

For nonzero vectors x, y in X, let $V(x, y)$ denote the subspace of X generated by x and y. A linear 2-normed space (X, v) is said to be strictly convex ([3]) if $v(x+y, z)=v(x, z)+v(y+z)$ and $z\notin V(x, y)$ imply that $y=ax$ for some $a>0$. Some characterizations of strict convexity for linear 2-normed spaces are given in [1, 3, 4, 5, 12]. Also, a linear 2-normed space (X, v) is said to be strictly 2-

*This research is supported by the Korean Ministry of Education Scholarship Foundation, 1983.

convex ([6]) if $v(x, y)=v(x, z)=v(y, z)=1/3v(x+z, y+z)=1$ implies that $z=x+y$. These spaces have been studied in [2, 4, 6, 13]. It is easy to see that every strictly convex linear 2-normed space is always strictly 2-convex but the converse is not necessarily true. Throughout this paper, let (X, v) denote a linear 2-normed space.

I. Strict convexity and strict 2-convexity.

In this section, we give some characterizations of strict convexity and strict 2-convexity for linear 2-normed spaces. The following theorem is well-known:

THEOREM 1.1 ([12]). *The following statements are equivalent:*
(1) (X, v) *is strictly convex.*
(2) $v(x, z)=v(y, z)=1$ *and* $z\notin V(x, y)$ *imply that* $v(1/2(x+y), z)<1$.

We will say that a linear 2-normed space (X, v) has the property (P) if for any x, y, z in X with $x\neq y$, $v(y, z)=v(y, z)=1$ and $z\notin V(x, y)$, we have $v(ax+(1-a)y, z)<1$ for some real number a.

If M and N are subspaces of X, then a bilinear functional F on $M\times N$ is said to be bounded if there exists a number $K>0$ such that for every $(x, y)\in M\times N$, $|F(x, y)|\leq Kv(x, y)$. The norm of F, $\|F\|$, is defined by $\|F\|=\inf \{K; |F(x, y)|\leq Kv(x, y)$ for every (x, y) in $M\times N\}$. Additional informations about bounded bilinear functionals and the Hahn-Banach theorem type on linear 2-normed spaces may be found in [7, 8, 13].

THEOREM 1.2. *The following statements are equivalent:*
(1) (X, v) *is strictly convex.*
(2) (X, v) *has the property* (P).

Proof. Clearly if (X, v) is strictly convex, then, by Theorem 1.1. the property (P) holds with $a=1/2$, so it is enough to prove the converse. For the converse, it is sufficient to prove that if (X, v) is not strictly convex, then (X, v) does not have the property (P). Suppose that x, y, z in X with $x\neq y$, $v(x, z)=v(y, z)=v(1/2(x+y), z)=1$, and $z\notin V(x, y)$. By Hahn-Banach theorem type of [13], there exists a bounded bilinear functional F defined on $X\times V(z)$ such that $\|F\|=1$ and $F(1/2(x+y), z)=v(1/2(x+y), z)=1$. So we have $1/2F(x, z)+1/2F(y, z)=1$ and $F(x, z)\leq|F(x, z)|\leq\|F\|v(x, z)=1$, $F(y, z)\leq|F(y, z)\leq\|F\|v(y, z)=1$. From these properties, it follows that $F(x, z)=F(y, z)=1$. Therefore, for any real number a, $v(ax+(1-a)y, z)\geq|F(ax+(1-a)y, z)|=|aF(x, z)+(1-a)F(y, z)|=1$, and so the property (P) fails.

A point p of X is called a 2-norm midpoint of 3 non-collinear points a, b, c in X if $d(a, b, p)=d(a, p, c)=d(p, b, c)=1/3d(a, b, c)$. For non-collinear a, b, c in X, let $T(a, b, c)=\{x\in X;\ d(a, b, c)=d(a, b, x)+d(a, x, c)+d(x, b, c)\}$. $T(a, b, c)$ will be called the triangle with vertices a, b and c. Furthermore, we will designate the area of $T(a, b, c)$ to be $d(a, b, c)$. A point p of X will be a center of $T(a,$

$b, c)$ if p is a 2-norm midpoint of a, b and c. From [6], $T(a, b, c)$ has a unique center if and only if (X, v) is strictly 2-convex, and for non-collinear a, b, c, $T(a, b, c)$ is convex. If a, b and c are non-collinear, let $C(a, b, c)$ denote the convex envelope of $\{a, b, c\}$, i.e., $C(a, b, c)$ is the smallest convex set containing $\{a, b, c\}$. In particular, we will use the result that $C(a, b, c) = \{\alpha a + \beta b + \gamma c; \alpha, \beta, \gamma, \geq 0 \text{ and } \alpha + \beta + \gamma = 1\}$.

THEOREM 1.3 ([6]). *The following statements are equivalent:*

(1) (X, v) *is strictly 2-convex.*

(2) *If* a, b *and* c *are non-collinear, then* $T(a, b, c) = C(a, b, c)$.

(3) *If* a, b *in* X *with* $v(a, b) > 0$, *then there exists a unique point* c *in* X *such that* 0 *is a center of* $T(a, b, c)$.

(4) *If* $v(a, b) = v(a, c) = v(b, c) = 1$, $c \neq -(a+b)$, $\alpha, \beta, \gamma > 0$ *and* $\alpha + \beta + \gamma = 1$, *then* $d(\alpha a, \beta b, \gamma c) < 1/3$.

II. Continuity of linear mappings

In this section, we define continuity of linear mappings on linear 2-normed spaces and investigate their some properties.

We will say that a sequence $\{x_n\}$ in X converges weakly to a point x in X if $\lim_{n \to \infty} v(x_n - t, x - t) = 0$ for all t in x.

THEOREM 2.1. (1) $x \in T(a, b, c)$ *if and only if* $x - y \in T(a - y, b - y, c - y)$.

(2) $T(\alpha a, \alpha b, \alpha c) = |\alpha| T(a, b, c)$, *where* α *is non-zero real.*

(3) $T(a+p, b+p, c+p) = T(a, b, c) + p$.

(4) $T(a, b, c)$ *is weakly closed.*

Proof. we shall establish (2) and (4).

(2) If $x \in T(\alpha a, \alpha b, \alpha c)$, then $v(\alpha a - \alpha c, \alpha b - \alpha c) = v(\alpha a - x, \alpha b - x) + v(\alpha a - x, \alpha c - x) + v(\alpha b - x, \alpha c - x)$. Multiplying by $1/|\alpha|$, we have $x/|\alpha| \in T(a, b, c)$ or $x \in |\alpha| T(a, b, c)$. The other inclusion is similar. (4) Assume that $x_n \in T(a, b, c)$ and $\{x_n\} \longrightarrow x$ weakly as $n \to \infty$. We have

$$\begin{aligned}
& v(a - x_n, a - x) = v(a - x_n, -(b - x_n) + (b - x_n) + a - x) \\
& \geq v(a - x_n, b - x_n) - v(a - x_n, b - x_n + a - x) \\
& = v(a - x_n, b - x_n) - v(a - x_n, b - x) \\
& = v(a - x_n, b - x_n) - v(a - x_n + x - x, b - x) \\
& \geq v(a - x_n, b - x_n) - v(a - x, b - x) - v(x_n - x, b - x) \\
& = v(a - x_n, b - x_n) - v(a - x, b - x) - v(b - x_n, b - x).
\end{aligned}$$

Therefore, $v(a - x_n, a - x) + v(b - x_n, b - x)$

$$\geq v(a - x_n, b - x_n) - v(a - x, b - x).$$

In a similar fashion,

$$v(a - x_n, a - x) + v(b - x_n, b - x)$$

$$\geq v(a-x,\ b-x)-v(a-x_n,\ b-x_n).$$

From the definition of weak convergence, we have

$$v(a-x_n,\ b-x_n)\longrightarrow v(a-x,\ b-x).$$

Hence it now follows that $x\in T(a,b,c)$.

Let (X, v) and (X_1, v) be linear 2-normed spaces. A mapping $f: X\longrightarrow X_1$ is said to be continuous at p in X if for each $\varepsilon>0$, there is a $\delta>0$ such that if $v(a-c,\ b-c)<\delta$ and $p\in T(a,b,c)$, then $v(f(a)-f(c),\ f(b)-f(c))<\varepsilon$ and $f(T(a,b,c))\subset T(f(a),\ f(b),\ f(c))$.

THEOREM 2.2. *Let (X, v) and (X_1, v) be linear 2-normed spaces. If a mapping $f: X\longrightarrow X_1$ is linear and continuous at 0, then f is continuous at p.*

Proof. Given $\varepsilon>0$, there is a $\delta>0$ such that if $v(a-c,\ b-c)<\delta$ and $0\in T(a, b,c)$, then $v(f(a)-f(c),\ f(b)-f(c))<\varepsilon$ and $f(T(a,b,c))\subset T(f(a)),\ f(b), f(c))$. Assume that $v(x-z,\ y-z)<\delta$ and $p\in T(x,y,z)$. By Theorem 2. 1, $0\in T(x-p,\ y-p,\ z-p)$. Also, $v(x-p-z+p,\ y-p-z+p)<\delta$, $v(f(x)-f(z), f(y)-f(z))<\varepsilon$ and $f(T(x-p,\ y-p,\ z-p))\subset T(f(x-p),\ f(y-p),\ f(z-p))$. Let $q\in T(x,y,z)$. Then $q-p\in T(x-p,\ y-p,\ z-p)$. Hence $f(q-p)\in f(T(x-p,\ y-p,\ z-p))$, i.e., $f(q)-f(p)\in T(f(x)-f(p),\ f(y)-f(p),\ f(z)-f(p))$. Thus $f(q)\in T(f(x),\ f(y),\ f(z))$ Therefore, $f(T(x,\ y,\ z))\subset T(f(x),\ f(y), f(z))$, i.e., f is continuous.

COROLLARY 2.3. *Let (X, v) and (X_1, v) be linear 2-normed spaces. If a mapping $f: X\longrightarrow X_1$ is linear, then f is continuous at p if and only if f is continuous at 0.*

Let (X, v) and (X_1, v) be linear 2-normed spaces. We will say that a mapping $f: X\longrightarrow X_1$ is bounded if there is a number $K\geq 0$ such that $v(f(x)-f(y),\ f(z)-f(y))\leq Kv(x-y,\ z-y)$.

THEOREM 2.4. *Let (X,v) and (X_1,v) be linear 2-normed spaces. If a mapping $f: X\longrightarrow X_1$ is linear and continuous, then f is bounded.*

Proof. If $v(x-y,\ z-y)=0$, then $x-y=\alpha(z-y)$. Hence $f(x-y)=\alpha f(z-y)$ and therefore $v(f(x)-f(y),\ f(z)-f(y))=0$. Assume that $v(x-y,\ z-y)=\alpha\neq 0$. Since f is continuous at 0, there is a $\delta>0$ such that if $v(a-c,b-c)<\delta$ and $0\in T(a,b,c)$, then $v(f(a)-f(c),\ f(b)-f(c))<1$ and $f(T(a,b,c))\subset T(f(a),\ f(b), f(c))$. Let $\beta=(\delta/2\alpha)^{1/2}$. $v(\beta(x-y),\ \beta(z-y))=\delta/2<\delta$ and $0\in T(\beta(x-y),\ 0, \beta(z-y))$, $v(f(\beta(x-y)),\ f(\beta(z-y)))<1$. Hence $v(f(x)-f(y),\ f(z)-f(y))< 2/\delta\ v(x-y,\ z-y)$. Therefore f is bounded.

THEOREM 2.5. *Let (X, v) and (X_1, v) be linear 2-normed spaces. If a mapping $f: X\longrightarrow X_1$ is linear and bounded, and (X,v) is strictly 2-convex, then f is* *continuous.*

Proof. Assume that $v(f(a)-f(c),\ f(b)-f(c))<Kv(a-c,\ b-c)$ and $\varepsilon>0$. Let $\delta=\varepsilon/(K+1)$. If $v(x-y,\ y-z)<\delta$ and $0\in T(x,y,z)$, then $v(f(x)-f(z),\ f(y)-f(z))<Kv(x-z,\ y-z)<\varepsilon$. By Theorem 1.3., since $(X,\ v)$ is strictly 2-convex, either $C(x,y,z)=T(x,y,z)$ or $z=\alpha x+\beta y$, $\alpha+\beta=1$. Assuming the former, if $p\in T(x,y,z)$, then $p=\alpha x+\beta y+\gamma z$, α, β, $\gamma\geq 0$, $\alpha+\beta+\gamma=1$. Therefore, using the linearity, we have $f(p)\in T(f(x),\ -f(y),\ f(z))$, i.e., f is continuous. Assuming the latter, $z=\alpha x+\beta y$, $\alpha+\beta=1$, we have $v(x-z,\ y-z)=0$. Thus, if $p\in T(x,y,z)$, $v(x-p,\ y-p)=v(x-p,\ z-p)=v(y-p,\ z-p)=0$. Since f is bounded, $f(p)\in T(f(x),\ f(y),\ f(z))$, i.e., f is continuous.

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