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SUBELLIPTICITY AND THE COMMUTATORS

RAK JUNG KIM AND CHOON HO LEE

0. Introduction

Throughout this paper let Ω be a bounded open subset of R^n. For any real number s, we define

$$\|u\|_s^2=\int|\hat{u}(\xi)^2(1+|\xi|^2)^s d\xi \quad \text{for} \quad u\in C_0^\infty(\Omega)$$

and we denote by $H_s(\Omega)$ the completion of $C_0^\infty(\Omega)$ in this norm. We will consider the algebra $OPS^m(\Omega)$ of pseudodifferential operators of order m. To define pseudodifferential operators we use the formula

$$P(x, D)u(x)=\int e^{ix\cdot\xi}p(x, \xi)\hat{u}(\xi)d\xi \qquad (x\in\Omega)$$

Let $P(x, D)$ be a properly supported pseudodifferential operator of order m. Then its associated kernel $K(x, y)\in\mathcal{D}'(\Omega\times\Omega)$ has the following property.

Given any function $g\in C_0^\infty(\Omega)$, both $g(x)K(x, y)$ and $K(x, y)g(y)$ have compact supports (contained in $\Omega\times\Omega$).

If $P\in OPS(\Omega)$, and K is a compact subset of Ω, we have the estimate

$$\|Pu\|_s\leq C\|u\|_{m+s}, \qquad u\in C_0^\infty(K)$$

for all real numbers s.

Moreover, if K is a compact subset, for all real s, $P(x, D)$ induces a compact linear map $H_s(K)\to H_{s-m}(K)$.

Hörmander proved the inequality in theorem 5. In this paper we simplify the theorem in [1] by using the commutator.

1. Subellipticity

DEFINITION. A pseudodifferential operator p of order m will be called subelliptic if, for every real ρ, every compact set K in Ω, and every s', there is a constant C such that

$$\|u\|_{m+\rho-1/2}\leq C(\|Pu\|_\rho+\|u\|_{s'}), \quad u\in C_0^\infty(K).$$

When f and g are C^∞ functions on Ω, the Poisson bracket $\{f, g\}$ is defined by

$$\{f, g\}=\sum\frac{\partial f}{\partial x_j}\frac{\partial g}{\partial \xi_j}-\frac{\partial g}{\partial x_j}\frac{\partial f}{\partial \xi_j}.$$

Then $\frac{1}{i}\{p_m(x, \xi), \bar{p}_m(x, \xi)\}$ is the principal symbol of a commutator $C=[p^*, p]$ where $p_m(x,\xi)$ is the principal symbol of a pseudodifferential operator p, and

$C=p^*p-pp^*\in OPS^{2m-1}$ can be proved.

THEOREM 1. *Suppose* $P\in OPS^m(K)$ *and* $c_{2m-1}=\frac{1}{i}\{p_m(x,\ \xi)\},\ \bar{p}_m(x,\ \xi)\}>0$ *on a compact subset* K *of* Ω *whenever* $p_m(x,\ \xi)=0$. *Then, for every real* m, *there exists a positive constant* C, *dependent on* m, *such that*

$$\|u\|^2_{m-1/2}\leq C(\|Pu\|_0{}^2+\|u\|^2_{m-1})\qquad u\in C_0^\infty(K)$$

Proof. Let $C=p^*p-pp^*$. Observe that $p(x,\ \xi)\sim p_m(x,\ \xi)$ for ξ large enough. From the hypothesis c_{2m-1}=principal symbol of $C>0$ whenever

$$p_m(x,\ \xi)=0$$

and from the fact that $\eta|\xi|^{-1}|p_m(x,\ \xi)|^2+c_{2m-1}\geq 0$ for ξ large enough if $p_m(x,\ \xi)\neq 0$, and $\eta>0$ large enough, $\eta\Lambda^{-1}p^*p+[p^*,\ p]$ is strongly elliptic where Λ is the operator defined by $\widehat{\Lambda u}(\xi)=(1+|\xi|^2)^{1/2}\hat{u}(\xi)$.
Since

$$\begin{aligned}(p^*pu,\ u)&=(pp^*u,\ u)+(Cu,\ u)\\ &\geq(Cu,\ u),\end{aligned}$$

we have that

$$\begin{aligned}2\|pu\|_0{}^2&=\|pu\|_0{}^2+(p^*pu,\ u)\\ &\geq\|pu\|_0{}^2+(Cu,\ u).\end{aligned}$$

Recall that, for three real numbers s, s', s'' such that $s''<s'<s$, there is a constant $c_\varepsilon>0$ such that

$$\|u\|_{s'}\leq\varepsilon\|u\|_s+c_\varepsilon\|u\|_{s''},\ \ u\in H_s(K)$$

to every number $\varepsilon>0$.
For some $k>0$

$$\begin{aligned}\|pu\|^2_{-1/2}&\leq\frac{1}{\eta}\|pu\|_0{}^2+k\|pu\|_{-1}{}^2\\ &\leq\frac{1}{\eta}\|pu\|_0{}^2+k'\|u\|^2_{m-1}.\end{aligned}$$

Thus

$$\|pu\|_0{}^2\geq\eta\|pu\|^2_{-1/2}-k'\|u\|^2_{m-1}\qquad(k'>0).$$

Taking

$$A=C+\eta p^*\Lambda^{-1}p$$

we have

$$\begin{aligned}2\|pu\|_0{}^2&\geq(Au,\ u)-k''\|u\|_{-1}{}^2\qquad(k''>0)\\ &\geq k_1\|u\|^2_{m-1/2}-k_2\|u\|^2_{m-1}\quad\text{for some positive}\end{aligned}$$

constants k_1 and k_2, by the Gårding's Inequality.

Substituting $\Lambda^\rho u$ for u we obtain $\|u\|_{m+\rho-1/2}\leq c(\|pu\|_\rho+\|u\|_{m+\rho-1})$. We deduce from theorem 1 the following consequences.

THEOREM 2. *Suppose that* $P\in OPS^m(K)$ *and suppose that* $c_{2m-1}=\frac{1}{i}\{p_m,\ \bar{p}_m\}$

>0 *whenever* $p_m(x,\xi)=0$. *If* $u\in\mathcal{D}'(K)$ *and* $pu\in H_\rho(K)$, *then* $u\in H_{m+\rho-1/2}(K)$ *for every real* ρ. *In addition, the set* $\{u\,|\,u\in\mathcal{D}'(K)\cap Ker\ P\}$ *is a finite dimensional subspace of* $C^\infty(K)$.

Proof. The first assertion can be shown from the fact that $\mathcal{E}'(\Omega)=\bigcup_s H_s^c(\Omega)$. Suppose now that $u\in\mathcal{D}'(K)$ and $Pu\in H_\rho(K)$. Since K is compact, $u\in H_{s'}(K)$ for some real s' If $s'\geq m+\rho-1$, then $u\in H_{m+\rho-1}(K)$ and $Pu\in H_\rho(K)$ imply that $u\in H_{m+\rho-1/2}$. If $s'<m+\rho-1$, by Korn's lemma, $u\in H_{m+\rho-1/2}$. That $u\in C^\infty(K)$ follows immediately from the fact that $C^\infty(K)\supset\bigcap_s H_s(K)$. Letting $N=\{u\,|\,u\in\mathcal{D}'(K)\ Pu=0\}$, we obtain from theorem 1 that

$$\|u\|_{m-1/2}\leq c\|u\|_{m-1}.$$

Recall that the natural embedding $i: H_s(K)\hookrightarrow H_{s-\delta}(K)$ is compact for all real s and $\delta>0$ where K is compact. Thus, setting $s=m-1/2$ and $\delta=\frac{1}{2}$, the set $i\{u|\ Pu=0,\ \|u\|_{m-1/2}\leq 1\}$ becomes a compact subset of $H_{m-1}\cap\mathrm{Ker}\,P$. But this is possible if and only if N is finite dimensional.

THEOREM 3. *Let all hypotheses be satisfied as those of above theorem. If* $f\in H_s(K)$ *satisfies*

$$(f,\ u)=0 \quad \text{for all} \quad u\in C^\infty(K) \quad \text{and} \quad u\in Ker\ P.$$

Then there exists $v\in H_{s+m-1/2}(K)$ *such that* $P^*v=f$.

Let $P\in OPS^m(\Omega)$ be a properly supported pseudodifferential operator. Let $0\in\Omega$ and let ω be a C^∞ function in a neighborhood of 0 with a Taylor expansion of the form

$$\begin{aligned}\omega(x)&=\langle x,\ \xi\rangle+\omega_2(x)+\omega_3(x)\\&=\langle x,\ \xi\rangle+\omega_1(x)\end{aligned}$$

where $R_n\ni\xi\neq0$, ω_2 is a homogeneous second degree polynomial with Im ω_2 positive definite and $\omega_3(x)=0(|x|^3)$, $x\to0$. Let Ω' be a neighborhood of 0 where

$$\mathrm{Im}\ \omega_2-|\omega_3|\geq c|x|^2$$

where c is a positive constant, and assume that Ω' is relatively compact in Ω. We denote the symbol of P by $\sum_0^\infty p_{m-j}(x,\xi)$.

Not that $p^{(\alpha)}{}_{m-j}(x,\ \tau\xi)=0(\tau^{m-j-\alpha})$ and $D^\alpha(\phi e^{i\tau\omega_1})=0(\tau^{\frac{|\alpha|}{2}})$ for $\phi\in C_0^\infty(\Omega')$ where Ω' is a neighborhood of 0. The following Lemma is due to Hörmander [1]

LEMMA 4. *For every* $\phi\in C_0^\infty(\Omega')$ *we have*

$$P(\phi e^{i\tau\omega})\sim e^{i\tau\langle x,\xi\rangle}\sum_{\alpha,j}p^{(\alpha)}{}_{m-j}D^\alpha(\phi e^{i\tau\omega_1})/\alpha!,\quad \tau\to\infty$$

where the series is asymptotic in the sense that

$$P(\phi e^{i\tau\omega})-\sum_{|\alpha|<N}\sum_{j<J}p_{m-j}{}^{(\alpha)}(x,\ \tau\xi)D^\alpha(\phi e^{i\tau\omega_1})/\alpha!=0(\tau^{m-J}+\tau^{m-\frac{N}{2}})$$

THEOREM 5. *Let $P\in OPS^m(\Omega)$ and $0\neq\xi\in T_x^*$, the cotangent bundle over x, and assume that the condition.*

(**) $C_{2m-1}(x,\xi)=\frac{1}{i}\{p_m,\ \bar{p}_m\}<0$ *whenever* $p_m(x,\xi)=0$.

Then there is no estimate of the form

$$\|u\|_s\leq C(\|Pu\|_r+\|u\|_t)\quad u\in C_0^\infty(\Omega')$$

where Ω' is a neighborhood of x, and $t<s$.

Proof. It is sufficient to show that the estimate

$$\|u\|_\rho\leq C(\|pu\|_0+\|u\|_0)\qquad u\in C_0^\infty(\Omega')$$

cannot hold for any $\rho>0$, [1]. x may be assumed to be the origin Observe that

$$\|u\|_0\leq C\|u\|_\rho\qquad u\in C_0^\infty(\Omega')$$

for sufficiently small neighborhood Ω' of 0. We shall prove that

$$\|u\|_0\leq\|u\|_\rho\leq 2C\|Pu\|_0\qquad u\in C_0^\infty(\Omega')$$

is false. Take a function of the form

$$u_\tau=e^{i\tau\omega}\sum_0^{\sigma-1}\phi_j\tau^{-j}\qquad \phi_j\in C_0^\infty(\Omega')$$

with functions $\phi_0,\ \phi_1,\ \cdots,\ \phi_{\sigma-1}\in C_0^\infty(\Omega')$ to be determined. From the condition (**) $p_m^{(j)}(0,\xi)\neq 0$ for some j

Thus it is possible to choose a symmetric matrix $\frac{\partial^2\omega_2}{\partial x_j\partial x_k}$ having positive definite imaginary part, such that

$$\sum_j\frac{\partial^2\omega_2}{\partial x_j\partial x_k}p_m^{(j)}(0,\ \xi)+p_{m,(k)}(0,\ \xi)=0$$

where $p_m^{(j)}=\frac{\partial p_m}{\partial\xi_j}$, $p_{m,(k)}=\frac{\partial p_m}{\partial x_k}$. [2]

Following the proof of Lemma 6. 1. 3 [2], for any given in teger σ, we can choose a polynomial $\omega_1=\omega_2+\omega_3$ and $\omega=\langle x,\ \xi\rangle+\omega_1(x)$ where ω_2 is homogeneous of second order, Im ω_2 is positive definite, $|\omega_3(x)|=0(|x|^3)$, $x\to 0$, and

$$\sum_{|\alpha|<2\sigma}p_m^{(\alpha)}(x,\xi)\ \ (\text{grad }\omega_1)^\alpha/\alpha!=0(|x|^{2\sigma}),\ \ x\to 0.$$

Note that $\omega=\langle x,\xi\rangle+\omega_2+\omega_3=\langle x,\xi\rangle+\omega_1$

$\text{grad }\omega=\xi+\text{grad }\omega_1$

$p_m(x,\ \text{grad }\omega)=\sum_{|\alpha|<2\sigma}p_m^{(\alpha)}(x,\xi)\ \ (\text{grad }\omega_1)^\alpha/\alpha!+\sum_{|\alpha|\geq 2\sigma}$

Thus we have

$$\begin{aligned}\|u_\tau\|_0^2&=\int|e^{i\tau\omega}\sum_j^{\sigma-1}\phi_j\tau^{-j}|^2dx\\&=\tau^{-\frac{n}{2}}\int|e^{i\tau\omega\left(\frac{x}{\sqrt{\tau}}\right)}\sum_j^{\sigma-1}\phi_j\left(\frac{x}{\sqrt{\tau}}\right)\tau^{-j}|^2dx\\&\sim|\phi_0(0)|^2\tau^{-\frac{n}{2}}e^{-2\mathrm{Im}\omega_2}dx.\end{aligned}$$

By Lemma 4, since $p_{m-\sigma}(x,\tau\xi)=0(\tau^{m-\sigma})$,

$pu_\tau=e^{i\tau\langle x,\xi\rangle}\sum_{|\alpha|<2\sigma}\sum_{j<\sigma}\sum_{\nu<\sigma}p_{m-j}^{(\alpha)}(x,\tau\xi)D^\alpha(\phi_\nu e^{i\tau\omega_1})\tau^{-\nu}/\alpha!+0(\tau^{m-\sigma})$

Put $A=\sum_{|\alpha|<2\sigma}p_m^{(\alpha)}(x,\xi)(\text{grad }\omega_1)^\alpha/\alpha!=0(|x|^{2\sigma})$

$A_j = \sum_{|\alpha| < 2\sigma - 1} p_m^{(\alpha, j)}(x, \xi)$ (grad $\omega_1)^\alpha / \alpha!$

Then from the condition (**) $A_j(0) \neq 0$ for some j. Letting

$e^{i\tau\langle x, \xi\rangle} \sum_{|\alpha| < 2\sigma} \sum_{j<\sigma} \sum_{\nu<\sigma} p^{(\alpha)}{}_{m-j}(x, \tau\xi) D^\alpha(\phi_\nu e^{i\tau\omega_1}) \tau^{-\nu}/\alpha! = \tau^m e^{i\tau\omega} \sum_0^{4\sigma} a_\mu \tau^{-\mu}$,

since $p^{(\alpha)}{}_{m-j}(x, \tau\xi) = 0(\tau^{m-j-\alpha})$, $D^\alpha(\phi_\nu e^{i\tau\omega_1}) = 0(\tau^{\frac{|\alpha|}{2}})$

we have $a_0 = A\phi_0$,

$$a_1 = A\phi_1 + \sum_1^n A_j D_j \phi_0 + B\phi_0 \text{ for some } C^\infty \text{ function } B$$

The general form of the coefficients, if $\mu < \sigma$,

$$a_\mu = A\phi_\mu + \sum_1^n A_j D_j \phi_{\mu-1} + B\phi_{\mu-1} + L_u$$

where L_u is a linear combination of functions ϕ_ν with $\nu < \mu - 1$ and their derivatives. In view of the proof of Theorem 6.1.1 [2] we can choose $\phi_0, \phi_1, \cdots, \phi_{\sigma-1}$ so that $\phi_0(0) \neq 0$ and $a_\mu = 0(|x|^{2(\sigma-\mu)})$, $x \to 0$ since $A_j(0) \neq 0$, for some j. Shrinking Ω' if necessary we can make it valid in Ω'. Taylor expansion of

$$\begin{aligned} \omega(x) &= \langle x, \xi\rangle + \omega_1(x) \\ &= \langle x, \xi\rangle + \omega_2(x) + \omega_3(x) \end{aligned}$$

is given by

$$\omega_1(x) = \frac{1}{2} \sum \sum x^j x^k \alpha_{jk} + \omega_3(x)$$

where the matrix α_{jk} is symmetric and the matrix $\operatorname{Im} \alpha_{jk}$ is positive definite. Therefore

$$\begin{aligned} \operatorname{Im} \omega_2(x) - |\omega_3(x)| &\geq \frac{1}{2} \sum \sum x^j x^k \operatorname{Im} \alpha_{jk} \\ &\geq a|x|^2 \qquad (a > 0, \ x \in \Omega') \end{aligned}$$

By the Leibniz' formula,

$D^\alpha(a_\mu e^{i\tau\omega})$ = linear combination of terms of the form $D^\beta a_\mu D^{\alpha-\beta} e^{i\tau\omega}$ Moreover, since $D^\beta a_\mu = 0(|x|^{2(\sigma-\mu)-|\beta|})$

$$= 0(|x|^{2(\sigma-\mu-|\beta|)}), \quad x \to 0$$

and $D^{\alpha-\beta} e^{i\tau\omega} = e^{i\tau\omega} \tau^{|\alpha|-|\beta|}$ + terms with lower degree in τ. We have

$$\begin{aligned} |e^{i\tau\omega} D^\beta a_\mu| &= |e^{i\tau(\langle x, \xi\rangle + \omega_2(x) + \omega_3(x))} D^\beta a_\mu| \\ &= e^{-\tau \operatorname{Im}\omega_2 + \tau\omega_3} |D^\beta a_\mu| \\ &\leq e^{-\tau a|x|^2} 0(|x|^{2(\sigma-\mu-|\beta|)}), \\ &\leq e^{-\tau a|x|^2} (\tau|x|^2)^{\sigma-\mu-|\beta|} \tau^{-\sigma+\mu+|\beta|} \\ &\leq C\tau^{-\sigma+\mu+|\beta|} \end{aligned}$$

Note that $e^{-\tau a|x|^2}$ is a rapidly decreasing function with respect to $\tau|x|^2$ and

$$D^\alpha(a_\mu e^{i\tau\omega}) = 0(\tau^{-\sigma+\mu+|\alpha|})$$

Thus

$$\tau^m e^{i\tau\omega} a_\mu \tau^{-\mu} = 0(\tau^{m-\sigma}).$$

This implies that

$$\int |pu_\tau|^2 dx \leq C\tau^{2(m-\sigma)}$$

If σ is chosen large enough.

$$\lim \frac{\|u_\tau\|_0}{\|pu_\tau\|_0} = \infty,$$

which completes the Theorem.

THEOREM 6. *Assume that the condition* (**) *is valid for* $P\in OPS^m(\Omega)$ *and for* $\xi\neq 0$. *Then for any neighborhood* Ω' *of* x, *there exists* $f\in C_0^\infty(\Omega')$ *so that there is no* $u\in\mathcal{D}'(\Omega)$ *with* $Pu=f$ *in* Ω'.

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Seoul National University
Seoul 151, Korea