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ON THE RADIUS OF CONVEXITY OF ANALYTIC *P*-VALENT FUNCTIONS

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I. Introduction

Let $\mathcal{S}(\alpha, \beta)$ denote the class of functions

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n$$

which are analytic and univalent in the unit disk $\mathcal{U}=\{|z|<1\}$ and satisfy the condition

$$\left|\frac{f'(z)-\beta}{1-\beta}-\alpha\right|<\alpha \qquad (z\in\mathcal{U})$$

for $\alpha>1/2$ and $0\leqq\beta<1$.

This class $\mathcal{S}(\alpha, \beta)$ was studied by R.M. Goel and N.S. Sohi [3]. In particular, the class $\mathcal{S}(\alpha, 0)$ was studied by R.M. Goel [1], [2].

Let $\mathcal{S}_p(\alpha)$ denote the class of functions

$$f(z)=z^p+\sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p\in\mathcal{N})$$

which are analytic p-valent in the unit disk $\mathcal{U}$ and satisfy the condition

$$\left|\frac{f'(z)}{pz^{p-1}}-\alpha\right|<\alpha \qquad (z\in\mathcal{U})$$

for $\alpha>1/2$. This class $\mathcal{S}_p(\alpha)$ was studied by N.S. Sohi [4].

In this paper, we consider the analytic p-valent functions

$$f(z)=z^p+\sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p\in\mathcal{N})$$

in the unit disk $\mathcal{U}$ satisfying the condition

$$\left|\frac{f'(z)-\beta pz^{p-1}}{pz^{p-1}(1-\beta)}-\alpha\right|<\alpha \qquad (z\in\mathcal{U})$$

for $\alpha>1/2$ and $0\leqq\beta<1$. We denote the class of all such functions $f(z)$ by $\mathcal{S}_p(\alpha, \beta)$. The class $\mathcal{S}_1(\alpha, \beta)$ is the class $\mathcal{S}(\alpha, \beta)$ which was studied by R.M. Geol and N.S. Sohi [3] and the class $\mathcal{S}_p(\alpha, 0)$ is the class $\mathcal{S}_p(\alpha)$ which was studied by N.S. Sohi [4].

2. Radius of convexity for the class $S_p(\alpha, \beta)$

THEOREM. *Let a function*

$$f(z)=z^p+\sum_{n=1}^{\infty} a_{p+n}z^{p+n} \qquad (p\in\mathcal{N})$$

be in the class $\mathcal{S}_p(\alpha,\beta)$ *with* $1/2<\alpha\leqq 1$ *and* $0\leqq\beta\leqq 1/2$. *Then the function* $f(z)$ *is convex in*

$$|z|<\frac{\{p(1+A)-A+B\}-\sqrt{\{p(1+A)-A+B\}^2-4p^2A}}{2pA}$$

where $A=1/\alpha-1$ *and* $B=1-\beta+A\beta$.

Proof. Let

$$g(z)=\frac{f'(z)-\beta pz^{p-1}}{\alpha pz^{p-1}(1-\beta)}-1.$$

Then the function $g(z)$ has modulus at most 1 in the unit disk $\mathcal{U}$ and $g(0)=1/\alpha-1$. Further let

$$h(z)=\frac{g(z)-g(0)}{1-g(0)g(z)},$$

then $|h(z)|<1$ for $z\in\mathcal{U}$ and $h(0)=0$. Consequently, by using Schwarz's lemma, we have $h(z)=z\phi(z)$, where $\phi(z)$ is an analytic function in the unit disk $\mathcal{U}$ and satisfies $|\phi(z)|\leqq 1$ for $z\in\mathcal{U}$. Therefore we obtain

$$f'(z)=pz^{p-1}\frac{1+Bh(z)}{1+Ah(z)}$$
$$=pz^{p-1}\frac{1+Bz\phi(z)}{1+Az\phi(z)},$$

where $A=1/\alpha-1$ and $B=1-\beta+A\beta$. On differentiating both sides of the above equality with respect to z logarithmically, we get after some simplifications

$$1+\frac{zf''(z)}{f'(z)}=p+\frac{(B-A)\{z\phi(z)+z^2\phi'(z)\}}{\{1+Az\phi(z)\}\{1+Bz\phi(z)\}}.$$

Now, it is well-known that

$$|\phi'(z)|\leqq\frac{1-|\phi(z)|^2}{1-|z|^2}$$

for the analytic function $\phi(z)$ in the unit disk $\mathcal{U}$. Accordingly we have

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}\geqq p-\frac{(B-A)|z|\{|\phi(z)|+|z\phi'(z)|\}}{|\{1+Az\phi(z)\}\{1+Bz\phi(z)\}|}$$
$$\geqq p-\frac{(B-A)|z|\{|z|+|\phi(z)|\}\{1-|z\phi(z)|\}}{(1-|z|^2)|\{1+Az\phi(z)\}\{1+Bz\phi(z)\}|}.$$

Since

$$|\{1+Az\phi(z)\}\{1+Bz\phi(z)\}|\geqq\{1-A|z\phi(z)|\}\{1-B|z\phi(z)|\}$$

for $1/2<\alpha\leqq 1$ and $0\leqq\beta\leqq 1/2$, we get

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}\geqq p-\frac{(B-A)|z|\{|z|+|\phi(z)|\}\{1-|z\phi(z)|\}}{(1-|z|^2)\{1-A|z\phi(z)|\}\{1-B|z\phi(z)|\}}$$
$$\geqq p-\frac{(B-A)|z|}{(1-|z|)(1-A|z|)},$$

because $|\phi(z)|\leqq 1$ for $z\in\mathcal{U}$ and $0\leqq A<B$. The function $f(z)$ will be convex if and only if

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0,$$

which is satisfied if

$$pA|z|^2-\{p(1+A)-A+B\}|z|+p>0,$$

that is,

$$|z|<\frac{\{p(1+A)-A+B\}-\sqrt{\{p(1+A)-A+B\}^2-4p^2A}}{2pA}$$

Finally, let $\beta=0$ and $f(z)$ be defined by

$$f'(z)=pz^{p-1}\frac{1-z}{1-Az}$$

in which case, we can see that

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}=0$$

for

$$z=\frac{\{p(1+A)-A+1\}-\sqrt{\{p(1+A)-A+1\}^2-4p^2A}}{2pA}.$$

References

1. R.M. Goel, *A class of univalent functions whose derivatives have positive real part in the unit disc*, Nieuw Archief Voor Wiskunde, **15**(1967), 55–63.
2. R.M. Goel, *A class of analytic functions whose derivatives have positive real part in the unit disc*, Indian J. Math. **13**(1971), 141–145.
3. R.M. Goel and N.S. Sohi, *On a subclass of univalent functions*, Tamkang J. Math **10**(1979), 151–164.
4. N.S. Sohi, *A class of p-valent analytic functions*, Indian J. Pure Appl. Math. **10** (1979), 826–834.

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