ON THE RADIUS OF CONVEXITY OF ANALYTIC P-VALENT FUNCTIONS

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I. Introduction

Let \( \mathcal{D}(\alpha, \beta) \) denote the class of functions

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

which are analytic and univalent in the unit disk \( \mathcal{U} = \{ |z| < 1 \} \) and satisfy the condition

\[
\left| \frac{f'(z) - \beta}{1 - \beta} - \alpha \right| < \alpha \quad (z \in \mathcal{U})
\]

for \( \alpha > 1/2 \) and \( 0 \leq \beta < 1 \).

This class \( \mathcal{D}(\alpha, \beta) \) was studied by R.M. Goel and N.S. Sohi [3]. In particular, the class \( \mathcal{D}(\alpha, 0) \) was studied by R.M. Goel [1], [2].

Let \( \mathcal{D}_p(\alpha) \) denote the class of functions

\[
f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in \mathcal{U})
\]

which are analytic \( p \)-valent in the unit disk \( \mathcal{U} \) and satisfy the condition

\[
\left| \frac{f'(z)}{pz^{p-1}} - \alpha \right| < \alpha \quad (z \in \mathcal{U})
\]

for \( \alpha > 1/2 \). This class \( \mathcal{D}_p(\alpha) \) was studied by N.S. Sohi [4].

In this paper, we consider the analytic \( p \)-valent functions

\[
f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in \mathcal{U})
\]

in the unit disk \( \mathcal{U} \) satisfying the condition

\[
\left| \frac{f'(z) - \beta pz^{p-1}}{pz^{p-1}(1 - \beta)} - \alpha \right| < \alpha \quad (z \in \mathcal{U})
\]

for \( \alpha > 1/2 \) and \( 0 \leq \beta < 1 \). We denote the class of all such functions \( f(z) \) by \( \mathcal{D}_p(\alpha, \beta) \). The class \( \mathcal{D}_1(\alpha, \beta) \) is the class \( \mathcal{D}(\alpha, \beta) \) which was studied by R.M. Geol and N.S. Sohi [3] and the class \( \mathcal{D}_p(\alpha, 0) \) is the class \( \mathcal{D}_p(\alpha) \) which was studied by N.S. Sohi [4].
2. Radius of convexity for the class $S_p(\alpha, \beta)$

**Theorem.** Let a function

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n}z^{p+n}$$

($p \in \mathbb{N}$) be in the class $S_p(\alpha, \beta)$ with $1/2 < \alpha \leq 1$ and $0 \leq \beta \leq 1/2$. Then the function $f(z)$ is convex in

$$|z| < \frac{\{p(1+A) - A + B\} - \sqrt{\{p(1+A) - A + B\}^2 - 4p^2A}}{2pA}$$

where $A = 1/\alpha - 1$ and $B = 1 - \beta + A\beta$.

**Proof.** Let

$$g(z) = \frac{f'(z) - \beta p z^{p-1}}{\alpha p z^{p-1}(1-\beta)} - 1.$$

Then the function $g(z)$ has modulus at most 1 in the unit disk $\mathcal{U}$ and $g(0) = 1/\alpha - 1$. Further let

$$h(z) = \frac{g(z) - g(0)}{1 - g(0)g(z)},$$

then $|h(z)| < 1$ for $z \in \mathcal{U}$ and $h(0) = 0$. Consequently, by using Schwarz’s lemma, we have $h(z) = z\phi(z)$, where $\phi(z)$ is an analytic function in the unit disk $\mathcal{U}$ and satisfies $|\phi(z)| \leq 1$ for $z \in \mathcal{U}$. Therefore we obtain

$$f'(z) = pz^{p-1} \frac{1+Bh(z)}{1+Ah(z)}$$

$$= pz^{p-1} \frac{1+Bz\phi(z)}{1+Az\phi(z)},$$

where $A = 1/\alpha - 1$ and $B = 1 - \beta + A\beta$. On differentiating both sides of the above equality with respect to $z$ logarithmically, we get after some simplifications

$$1 + \frac{zf''(z)}{f'(z)} = p + \frac{(B-A) \{z\phi(z) + z^2\phi'(z)\}}{\{1+Az\phi(z)\} \{1+Bz\phi(z)\}}.$$

Now, it is well-known that

$$|\phi'(z)| \leq 1 - |\phi(z)|^2$$

for the analytic function $\phi(z)$ in the unit disk $\mathcal{U}$. Accordingly we have

$$\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq p - \frac{(B-A) \{|z| \{ |\phi(z)| + |z\phi'(z)| \} \}}{\{1+Az\phi(z)\} \{1+Bz\phi(z)\}}$$

$$\geq p - \frac{(B-A) \{|z| \{ |z| + |\phi(z)| \} |1 - z\phi(z)| \}}{(1-|z|^2) \{1+Az\phi(z)\} \{1+Bz\phi(z)\}}.$$

Since

$$\{1+Az\phi(z)\} \{1+Bz\phi(z)\} \geq \{1-A\phi(z)\} \{1-B\phi(z)\}$$

for $1/2 < \alpha \leq 1$ and $0 \leq \beta \leq 1/2$, we get
On the radius of convexity of analytic \( p \)-valent functions

\[
\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq p - \frac{(B-A) |z| [ |z| + |\phi(z)| ] [1 - |z\phi(z)|]}{(1-|z|^2) [1-A|z\phi(z)|] [1-B|z\phi(z)|]}
\geq p \frac{(B-A) |z|}{(1-|z|)(1-A|z|)},
\]

because \( |\phi(z)| \leq 1 \) for \( z \in \mathbb{U} \) and \( 0 \leq A < B \). The function \( f(z) \) will be convex if and only if

\[
\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0,
\]

which is satisfied if

\[
pA|z|^2 - \{p(1+A) - A+B\}|z| + p > 0,
\]

that is,

\[
|z| < \frac{\{p(1+A) - A+B\} - \sqrt{\{p(1+A) - A+B\}^2 - 4p^2A}}{2pA}
\]

Finally, let \( \beta = 0 \) and \( f(z) \) be defined by

\[
f'(z) = p z^{p-1} \frac{1-z}{1-Az}
\]

in which case, we can see that

\[
\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} = 0
\]

for

\[
z = \frac{\{p(1+A) - A+1\} - \sqrt{\{p(1+A) - A+1\}^2 - 4p^2A}}{2pA}.
\]

References

1. R.M. Goel, A class of univalent functions whose derivatives have positive real part in the unit disc, Nieuw Archief Voor Wiskunde, 15(1967), 55-63.
2. R.M. Goel, A class of analytic functions whose derivatives have positive real part in the unit disc, Indian J. Math. 13(1971), 141-145.

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