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A LEMMA ON AN INFINITESIMAL ETA-CONFORMAL TRANSFORMATION IN A COMPACT SASAKIAN MANIFOLD

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1. Introduction

A $(2n+1)$-dimensional differentiable manifold M is called to have a Sasakian structure if there is given a positive definite Riemannian metric g_{ji} and a triplet $(\phi_k{}^j, \xi^j, \eta_k)$ of $(1,1)$ type tensor field $\phi_k{}^j$, vector field ξ^j and 1-form η_k in M which satisfy the following equations

$$(1.1) \qquad \phi_j{}^i\phi_i{}^h=-\gamma_j{}^h,\ \ \phi_j{}^i\xi^j=0,\ \ \eta_i\phi_j{}^i=0,\ \ \eta_t\xi^t=1,$$
$$g_{st}\phi_j{}^s\phi_i{}^t=\gamma_{ji},\ \ \eta_i=g_{ih}\xi^h,$$

where

$$(1.2) \qquad \gamma_{ji}=g_{ji}-\eta_j\eta_i,\ \ \gamma_j{}^h=g^{ht}\gamma_{jt}$$

and

$$(1.3) \qquad \nabla_i\xi^h=\phi_i{}^h,\ \ \nabla_j\phi_i{}^h=-g_{ji}\xi^h+\delta_j{}^h\eta_i,$$

where ∇_k indicates the covariant differentiation with respect to g_{ji}. By virtue of the last equation of (1.1), we shall write η^h instead of ξ^h in the sequel. The indices $h, i, j, k, \ldots$ run over the range $\{1, 2, \ldots, 2n+1\}$.

In a $(2n+1)$-dimensional Sasakian manifold M, if an infinitesimal transformation v^h satisfies

$$(1.4) \qquad L_v g_{ji}=\lambda(g_{ji}+\eta_j\eta_i),$$

where λ is a scalar field and L_v denotes the Lie derivation with respect to v^h, then v^h is called an *infinitesimal eta-conformal transformation.*

Recently, K. Takamatsu and H. Mizusawa proved in [1] the following theorem on the case of dim $M>3$.

THEOREM. *If a compact Sasakian manifold M admits an infinitesimal eta-conformal transformation v^h defined by* (1.4), *then λ in* (1.4) *vanishes.*

The purpose of the present paper is to prove the theorem above stated on the case of dim $M=3$.

2. Proof of the theorem on the case of dim $M=3$

In a Sasakian manifold M, the following equations are satisfied (cf. [2]).

(2.1) $$\eta^k K_{kji}{}^t = \eta^t g_{ji} - \eta_i \delta_j^t,$$

and

(2.2) $$K_{kji}{}^t \eta_t = g_{ji}\eta_k - g_{ki}\eta_j,$$

where $K_{kji}{}^t$ is the curvature tensor of M.

From (2.1), we easily have

(2.3) $$\eta^k K_{kji}{}^t L_v \eta_t = g_{ji}\eta^t L_v \eta_t - \eta_i L_v \eta_j.$$

Substituting (1.4) into the formula (cf. [3])

$$L_v \left\{ {h \atop j\ i} \right\} = \frac{1}{2} g^{ht} (\nabla_j L_v g_{ti} + \nabla_i L_v g_{tj} - \nabla_t L_v g_{ji}),$$

where $\left\{ {h \atop j\ i} \right\}$ is the Christoffel symbol formed with g_{ji}, we get

(2.4) $$L_v \left\{ {h \atop j\ i} \right\} = \frac{1}{2} \{\lambda_j(\delta_i^h + \eta^h \eta_i) + \lambda_i(\delta_j^h + \eta_j \eta^h) - \lambda^h (g_{ji} + \eta_j \eta_i)\} + \lambda(\eta_j \phi_i{}^h + \eta_i \phi_j{}^h),$$

where we have put $\lambda_i = \nabla_i \lambda$ and $\lambda^h = g^{ht}\lambda_t$.

Substituting (2.4) into the formula (cf. [3])

$$L_v K_{kji}{}^h = \nabla_k L_v \left\{ {h \atop j\ i} \right\} - \nabla_j L_v \left\{ {h \atop k\ i} \right\},$$

we obtain

(2.5) $$\begin{aligned} L_v K_{kji}{}^h = \frac{1}{2} \{ & \phi_k{}^h \lambda_i \eta_j - \phi_j{}^h \lambda_i \eta_k + 2\phi_i{}^h (\lambda_k \eta_j - \lambda_j \eta_k) \\ & + (\nabla_k \lambda_i)(\delta_j^h + \eta^h \eta_j) - (\nabla_j \lambda_i)(\delta_k^h + \eta^h \eta_k) \\ & - (\nabla_k \lambda^h)(g_{ji} + \eta_j \eta_i) + (\nabla_j \lambda^h)(g_{ki} + \eta_k \eta_i) \\ & + \lambda^h (\eta_k \phi_{ji} - \eta_j \phi_{ki} - 2\eta_i \phi_{kj}) + \eta^h (\lambda_j \phi_{ki} \\ & - \lambda_k \phi_{ji} + 2\lambda_i \phi_{kj}) + \lambda_k \eta_i \phi_j{}^h - \lambda_j \eta_i \phi_k{}^h \\ & + 4\lambda \eta_i (\eta_j \delta_k^h - \eta_k \delta_j^h) + 2\lambda \eta^h (\eta_k g_{ji} - \eta_j g_{ki}) \\ & + 2\lambda (\phi_{ki} \phi_j{}^h - \phi_{ji} \phi_k{}^h + 2\phi_{kj} \phi_i{}^h)\}. \end{aligned}$$

Transvecting (2.5) with η_h, we obtain

(2.6) $$\begin{aligned} \eta_h L_v K_{kji}{}^h = \frac{1}{2} \{ & 2\eta_j \nabla_k \lambda_i - 2\eta_k \nabla_j \lambda_i - \eta_h (\nabla_k \lambda^h)(g_{ji} + \eta_j \eta_i) \\ & + \eta_h (\nabla_j \lambda^h)(g_{ki} + \eta_k \eta_i) + \lambda_j \phi_{ki} - \lambda_k \phi_{ji} + 2\lambda_i \phi_{kj} \\ & - \eta_h \lambda^h (\eta_j \phi_{ki} - \eta_k \phi_{ji} + 2\eta_i \phi_{kj}) + 2\lambda (\eta_k g_{ji} \\ & - \eta_j g_{ki})\}. \end{aligned}$$

Now taking the Lie derivative of both sides of (2.2), we have

(2.7) $$\begin{aligned} & K_{kji}{}^t L_v \eta_t + \eta_t L_v K_{kji}{}^t \\ & = g_{ji} L_v \eta_k - g_{ki} L_v \eta_j + \eta_k L_v g_{ji} - \eta_j L_v g_{ki}. \end{aligned}$$

Substituting (2.5) and (2.6) into (2.7) and transvecting it with η^k, we obtain

(2.8) $$\begin{aligned} \eta^k K_{kji}{}^t L_v \eta_t = g_{ji} \eta^k L_v \eta_k - \eta_i L_v \eta_j - \frac{1}{2} \{ & 2\eta^k \eta_j \nabla_k \lambda_i - 2\nabla_j \lambda_i \\ & - \eta^k \eta_t (\nabla_k \lambda^t)(g_{ji} + \eta_j \eta_i) + 2\eta_i \eta_t \nabla_j \lambda^t\}. \end{aligned}$$

Comparing (2.3) with (2.8), we obtain

$$2\eta^k\eta_j\nabla_k\lambda_i-2\nabla_j\lambda_i-\eta^k\eta_t(\nabla_k\lambda^t)(g_{ji}+\eta_j\eta_i)+2\eta_i\eta_t\nabla_j\lambda^t=0. \tag{2.9}$$

Transvecting (2.9) with g^{ji} and taking account of the fact that the dimension of M is equal to 3, we obtain $\nabla^i\lambda_i=0$, and from which

$$g^{ji}\nabla_j\nabla_i\lambda=0 \tag{2.10}$$

by virtue of $\lambda_i=\nabla_i\lambda$.

Thus, in a 3-dimensional compact Sasakian manifold M, we see that (cf. [3])

$$\lambda=\text{const.} \tag{2.11}$$

in whole manifold M.

On the other hand, we obtain

$$2\nabla_t v^t=g^{ji}L_v g_{ji}=4\lambda$$

by virtue of (1.4). Then by Green's theorem and (2.11), we see that

$$\lambda=0.$$

Thus we are done.

References

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