A LEMMA ON AN INFINITESIMAL ETA–CONFORMAL TRANSFORMATION IN A COMPACT SASTAKIAN MANIFOLD

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1. Introduction

A (2n+1)–dimensional differentiable manifold M is called to have a Sasakian structure if there is given a positive definite Riemannian metric g_{ji} and a triplet (φ^i_j, ζ^i_j, η^i_j) of (1,1) type tensor field φ^i_j, vector field ζ^i_j and 1–form η^i_j in M which satisfy the following equations

(1.1) φ^i_jφ^j_i = -γ^i_j, φ^i_jζ^i_j = 0, η^i_jφ^j_i = 0, η^i_jζ^j_i = 1,

where

(1.2) γ^i_j = g^i_j - η^j_iη^i_j, γ^h_j = g^h_iγ^i_j

and

(1.3) \nabla_iζ^h_i = φ^h_i, \nabla_jφ^h_i = -g^h_iζ^j_i + η^h_iζ^i_j,

where \nabla_k indicates the covariant differentiation with respect to g_{ji}. By virtue of the last equation of (1.1), we shall write η^h instead of ζ^h in the sequel. The indices h, i, j, k, ... run over the range \{1, 2, ..., 2n+1\}.

In a (2n+1)–dimensional Sasakian manifold M, if an infinitesimal transformation v^h satisfies

(1.4) L_vg_{ji} = \lambda (g_{ji} + η^j_iη^i_j),

where \lambda is a scalar field and L_v denotes the Lie derivation with respect to v^h, then v^h is called an infinitesimal eta–conformal transformation.

Recently, K. Takamatsu and H. Mizusawa proved in [1] the following theorem on the case of dim M > 3.

THEOREM. If a compact Sasakian manifold M admits an infinitesimal eta–conformal transformation v^h defined by (1.4), then \lambda in (1.4) vanishes.

The purpose of the present paper is to prove the theorem above stated on the case of dim M = 3.

2. Proof of the theorem on the case of dim M = 3

In a Sasakian manifold M, the following equations are satisfied (cf. [2]).

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\[
\eta^k K_{kji}' = \eta^i g_{ji} - \eta_i \delta_j',
\]
and
\[
K_{kji}' \eta_i = g_{ji} \eta_k - g_{ki} \eta_j,
\]
where \(K_{kji}'\) is the curvature tensor of \(M\).

From (2.1), we easily have
\[
\eta^k K_{kji}' L_v \eta_i = g_{ji} \eta^k L_v \eta_i - \eta_i L_v \eta_j.
\]
Substituting (1.4) into the formula (cf. [3])
\[
L_v \left\{ \frac{h}{i} \right\} = \frac{1}{2} g^{ht} (\nabla_i L_v g_{ti} + \nabla_t L_v g_{ti} - \nabla_t L_v g_{ti}),
\]
where \(\left\{ \frac{h}{i} \right\}\) is the Christoffel symbol formed with \(g_{ji}\), we get
\[
L_v \left\{ \frac{h}{i} \right\} = \frac{1}{2} \left\{ \lambda_j (\delta^h_i + \eta^h \eta_i) + \lambda_i (\delta^h_j + \eta_j \eta^h) - \lambda^h (g_{ji} + \eta_j \eta_i) \right\} + \lambda (\eta_j \phi_i - \eta_i \phi_j^h),
\]
where we have put \(\lambda_i = \nabla_i \lambda\) and \(\lambda^h = g^{ht} \lambda_t\).

Substituting (2.4) into the formula (cf. [3])
\[
L_v K_{kji}' = \nabla_k L_v \left\{ \frac{h}{i} \right\} - \nabla_j L_v \left\{ \frac{h}{k} \right\},
\]
we obtain
\[
L_v K_{kji}' = \frac{1}{2} \left\{ \phi_i^h \delta^j_k - \phi_j^h \delta^i_k - \phi_k^h \lambda^j_i - \phi_i^h \lambda^k_j \right\}
\]
\[
+ \lambda^h (g_{ki} + \eta_j \eta_i) + \lambda_i (\delta^h_j + \eta_j \eta^h) - \lambda^h (g_{ji} + \eta_j \eta_i),
\]
Transvecting (2.5) with \( \eta_k \), we obtain
\[
\eta_k L_v K_{kji}' = \frac{1}{2} \left\{ (\nabla_i L_v \lambda^h) (g_{ji} + \eta_j \eta_i) + \lambda_i (\delta^h_j + \eta_j \eta^h) - \lambda^h (g_{ji} + \eta_j \eta_i),
\]
\[
\eta_k \lambda^h (g_{ji} + \eta_j \eta_i) + \lambda_i (\delta^h_j + \eta_j \eta^h) - \lambda^h (g_{ji} + \eta_j \eta_i),
\]
Now taking the Lie derivative of both sides of (2.2), we have
\[
K_{kji}' L_v \eta_i + \eta_i L_v K_{kji}'
\]
\[
= g_{ji} L_v \eta_k - g_{ki} L_v \eta_j + \eta_k L_v g_{ji} - \eta_j L_v g_{ki},
\]
Substituting (2.5) and (2.6) into (2.7) and transvecting it with \( \eta^k \), we obtain
\[
\eta^k K_{kji}' L_v \eta_i = g_{ji} \eta^k L_v \eta_k - \eta_i L_v \eta_j - \frac{1}{2} \left\{ (\nabla_i L_v \lambda^h) (g_{ji} + \eta_j \eta_i) + \lambda_i (\delta^h_j + \eta_j \eta^h) - \lambda^h (g_{ji} + \eta_j \eta_i),
\]
\[
- \eta^k \nabla_i \eta_j (g_{ji} + \eta_j \eta_i) + \eta_j \eta^h \nabla_i \lambda_i \right\}.
\]
Comparing (2.3) with (2.8), we obtain

\[(2.9) \quad 2\eta^k\eta_i\nabla_k\lambda_i - 2\nabla_j\lambda_i - \eta^k\eta_i(\nabla_k\lambda_i)(g_{ji} + \eta_j\eta_i) + 2\eta_i\eta_i\nabla_j\lambda_i = 0.\]

Transvecting (2.9) with \(g^{ji}\) and taking account of the fact that the dimension of \(M\) is equal to 3, we obtain \(\nabla^i\lambda_i = 0\), and from which

\[(2.10) \quad g^{ji}\nabla_j\lambda_i = 0\]

by virtue of \(\lambda_i = \nabla_i\lambda\).

Thus, in a 3-dimensional compact Sasakian manifold \(M\), we see that (cf. [3])

\[(2.11) \quad \lambda = \text{const.}\]

in whole manifold \(M\).

On the other hand, we obtain

\[2\nabla_i\nu^i = g^{ji}L_\nu g_{ji} = 4\lambda\]

by virtue of (1.4). Then by Green's theorem and (2.11), we see that \(\lambda = 0\).

Thus we are done.

References


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