Bull. Korean Math. Soc. **21**(1984), No. 1, 21-26

NEARLY SASAKIAN MANIFOLD SATISFYING A CERTAIN CURVATURE CONDITION

CHONG-HŎN KIM AND BYŎNG-DU KIM

0. Introduction

The notion of nearly Sasakian manifold was introduced in [1] and Z-Olszak has studied certain properties in [2] and [3].

In section (2) of this paper, we show that a nearly Sasakian manifold M admitting $\Gamma_{ji}{}^h$ such that $\nabla_1 R_{kji}{}^h=0$ is of constant scalar curvature and the covariant derivate of the Ricci tensor of M is a symmetric tensor.

In the last section, we shall deal with a recurrent and conformal recurrent nearly Sasakian manifold.

1. Preliminaries

Let M be a $(2n+1)$-dimensional almost contact metric manifold, i.e., a Riemannian manifold together with a metric tensor g and tensor field F of type (1.1) and v a vector field, which satisfy

$$F_a{}^h F_j{}^a=-\delta_j^h+v_j v^h,\quad v_j=g_{ja}v^a,\ v_a v^a=1,$$
$$F_j{}^a F_i{}^b g_{ab}=g_{ji}-v_j v_i,\ F_a{}^i v^a=0.$$

It is well-known that in this condition the tensor field $F_{ji}=F_j{}^a g_{ai}$ is skew-symmetric.

The almost contact metric manifold M is called *a nearly Sasakian manifold* if

$$\nabla_k F_{ji}+\nabla_j F_{ki}=-2g_{kj}v_i+g_{ki}v_j+g_{ji}v_k, \tag{1.1}$$

where ∇ denotes the Riemannian connection with respect to the Christoffel symbols formed with g_{ji}.

Moreover, M satisfies

$$\nabla_k F_{ji}=-g_{kj}v_i+g_{ki}v_j$$

then it is called *Sasakian*, Thus, every Sasakian manifold is nearly Sasakian. The converse statement fails in general.

Supposing that M is nearly Sasakian manifold, the vector field v^h is a killing, that is, [1]

$$\nabla_j v_i+\nabla_i v_j=0. \tag{1.2}$$

We define the skew-symmetric tensor field H_{ji} by putting

$$\nabla_j v_i=F_{ji}+H_{ji}. \tag{1.3}$$

A nearly Sasakian manifold M is Sasakian if and only if M satisfies the condition $H_{ji}=0$. [2]

Set
$$H_j{}^i=H_{ja}g^{ai},\ H^{ji}=g^{ja}H_a{}^i,$$
$$F^{ji}=g^{ja}F_a{}^i.$$

In a nearly Sasakian manifold M, the following relations holds [3].

(1.4) $$(\nabla_j F_{ia})v^a=-g_{ji}+v_jv_i-H_{ja}F_i{}^a,$$

(1.5) $$H_{ja}F_i{}^a+H_{ia}F_j{}^a=0,$$

(1.6) $$H_a{}^iv^a=0,$$

(1.7) $$K_{akji}v^a=-\nabla_k F_{ji}-\nabla_k H_{ji}$$
$$=(g_{kj}+H_{ka}H_j{}^a)v_i-(g_{ki}+H_{ka}H_i{}^a)v_j,$$

(1.8) $$K_{ai}v^a=(2n+H_{ba}H^{ba})v_i,\quad H_{ba}H^{ba}=\text{constant},$$

(1.9) $$v^a\nabla_a K_{kjih}=0,$$

(1.10) $$K_{ja}F_i{}^a+K_{ia}F_j{}^a=0,$$

where K_{kjih} and K_{ji} are the curvature tensor and the Ricci tensor of M, respectively.

In a nearly Sasakian manifold, the following theorems are well known([3]).

THEOREM A. *If M is conformally flat, then* dim $M=5$ *and M is of constant curvature.*

THEOREM B. *If M satisfies $\nabla_m\nabla_l K_{kjih}-\nabla_l\nabla_m K_{kjih}=0$, then* dim $M=5$ *and M is of constant curvature.*

2. Semi-symmetric metric connection

In a Reimannian manifold M with a metric g_{ji}, we can take a semisymmetric metric connection with connection coefficient $\Gamma_{ji}{}^h$ [4] given by

$$\Gamma_{ji}{}^h=\left\{{h \atop j\ i}\right\}+\delta_j{}^hp_i-g_{ji}p^h,$$

where $\left\{{h \atop j\ i}\right\}$ are Christoffel symbols of M, P_i is a gradient vector and $p^h=g^{hk}p_k$.

The curvature tensor $R_{kji}{}^h$ of $\Gamma_{ji}{}^h$ and $K_{kji}{}^h$ of $\left\{{h \atop j\ i}\right\}$ are related in such a way that

(2.1) $$R_{kji}{}^h=K_{kji}{}^h-A_{ji}\delta_k{}^h+A_{ki}\delta_j{}^h-g_{ji}A_k{}^h+g_{ki}A_j{}^h,$$

where A_{ji} is a tensor of type (0.2) defined by

$$A_{ji}=\nabla_jp_i-p_jp_i+\frac{1}{2}g_{ji}p_kp^k,\quad\text{and}\quad A_k{}^h=g^{hl}A_{kl}.$$

We consider a nerarly Sasakian manifold M admitting $\Gamma_{ji}{}^h$ such that $\nabla_1R_{kji}{}^h=0$ Then we have

(2.2) $$\nabla_1K_{kji}{}^h-\nabla_1A_{ji}\delta_k{}^h+\nabla_1A_{ki}\delta_j{}^h-g_{ji}\nabla_1A_k{}^h+g_{ki}\nabla_1A_j{}^h=0.$$

Contracting with respect to h and k we get

(2.3) $$\nabla_1K_{ji}-(2n-1)\nabla_1A_{ji}-g_{ji}\nabla_1B=0,$$

where $B=g^{ab}A_{ab}$. Transvecting (2.3) with g^{ji}, we have

(2.4) $$\nabla_1 B=\nabla_1 K/4n,$$

where $K=g^{ji}K_{ji}$. Substituting (2.4) into (2.3), we get

(2.5) $$\nabla_1 A_{ji}=\{\nabla_1 K_{ji}-g_{ji}\nabla_1 K/4n\}/(2n-1),$$

from which

(2.6) $$\nabla_1 A_j{}^1=\{\nabla_1 K_j{}^1-\nabla_j K/4n\}/(2n-1).$$

Contracting with respect to 1 and h in (2.2), and using (2.5), (2.6) and the well-known relations

(2.7) $$\nabla_1 K_{kji}{}^1=\nabla_k K_{ji}-\nabla_j K_{ki},\quad \nabla_l K_k{}^l=\frac{1}{2}\nabla_k K,$$

we obtain that

(2.8) $$\nabla_k K_{ji}-\nabla_j K_{ki}=(g_{ji}\nabla_k K-g_{ki}\nabla_j K)/4n.$$

From (1.9), we find

(2.9) $$v^a\nabla_a K_{ji}=0,\ v^a\nabla_a K=0.$$

Moreover, transvecting (2.8) with v^k and using (2.9), we obtain

(2.10) $$v^a\nabla_j K_{ai}=\frac{1}{4n}v_i\nabla_j K.$$

On the other hand, we get

$$\begin{aligned} v^a\nabla_j K_{ai}+K_{ai}\nabla_j v^a &=\nabla_j(K_{ai}v^a)\\ &=(2n+H_{ba}H^{ba})\nabla_j v_i, \end{aligned}$$

from which

(2.11) $$v^a\nabla_j K_{ai}=(2n+H_{ba}H^{ba})(F_{ji}+H_{ji})-K_{ai}(F_j{}^a+H_j{}^a)$$

by virtue of (1.3) and (1.8).

Substituting (2.11) into (2.10), and contracting with v^i, we have

$$-(2n+H_{ba}H^{ba})v_a(F_j{}^a+H_j{}^a)=\frac{1}{4n}\nabla_j K.$$

Consquently, we get

(2.12) $$\nabla_j K=0.$$

Thus we have

THEOREM 2.1. *A nearly Sasakian manifold M admitting Γ_{ji}^h such that $\nabla_l R_{kji}{}^h=0$ is of constant scalar curvature.*

Since K is constant, from (2.8), we have $\nabla_k K_{ji}-\nabla_j K_{ki}=0$. That is, the tensor $\nabla_k K_{ji}$ is a symmetric tensor. Thus we have

THEOREM 2.2. *In a nearly Sasakian manifold M admitting Γ_{ji}^h such that $\nabla_l R_{kji}{}^h=0$, the covariant derivate of the Ricci tensor of M is a symmetric tensor.*

3. Some theorems on nearly Sasakian manifolds

We assume that a nearly Sasakian manifold has the vanishing Ricci tensor K_{ji}. Then, from (1.7), we have

$$(2n+H_{ba}H^{ba})v_i=0,$$

which is in consistent with such an assumption as the vector v_i is a unit vector.

Similarly, if we assume that a nearly Sasakian manifold is a locally flat, i.e., $K_{kji}{}^h=0$, then we have

$$(2n+H_{ba}H^{ba})v_i=0$$

which, by virtue of (1.8).

Thus we have

LEMMA 3.1. *In a nearly Sasakian manifold, the Ricci curvature tensor does not vanish. Especially, a nearly Sasakian manifold is not flat.*

Let M be *a recurrent nearly Sasakian manifold,*
i.e.,

$$\nabla_m K_{kjih}=x_m K_{kjih}, \tag{3.1}$$

where x_m is a non-zero covariant vector.

From (3.1), we get

$$\nabla_m \nabla_l K l_{jih}-\nabla_l \nabla_m K_{kjih}=(\nabla_m x_l-\nabla_l x_m)K_{kjih}, \tag{3.2}$$

Thus we have from Theorem B, Lemma 3.1 and (3.2)

THEOREM 3.2. *A necessary and sufficient condition for a recurrent nearly Sasakian manifold to be a space of constant curvature is that the recurrent vector is a gradient one.*

Let a tensor field $C_{kji}{}^h$ be the Weyl conformal curvature tensor in an almost contact metric manifold.

Then, by definition [6], $C_{kji}{}^h$ is given by

$$C_{kji}{}^h=K_{kji}{}^h+\delta_k^h L_{ji}-\delta_j^h L_{ki}+L_k{}^h g_{ji}-L_j{}^h g_{ki}, \tag{3.3}$$

where

$$L_{ji}=-\frac{K_{ji}}{2n-1}+\frac{Kg_{ji}}{4n(2n-1)}.$$

We shall consider a conformally recurrent nearly Sasakian manifold, that is,

$$\nabla_m C_{kji}{}^h=x_m C_{kji}{}^h \tag{3.4}$$

for a certain non-zero covariant vector field x_m.

In view of (1.9), we get

$$v^a\nabla_a K_{ji}=0,\quad v^a\nabla_a K=0, \tag{3.5}$$

from which

$$v^a\nabla_a L_{ji}=0. \tag{3.6}$$

Differentiating (3.3) covariantly and transvecting with v^m, we get

$$v^a x_a C_{kjih}=0, \tag{3.7}$$

by virtue of (3.4), (3.5) and (3.6).

From Theorem A and (3.7), we have

THEOREM 3.3. *The conformal recurrent nearly Sasakian manifold M is one of the following cases;*

(I) *M is of constant curvature and* dim $M=5$,

(II) *the structure vector is orthogonal to the recurrent vector.*

Next we assume that the following condition satisfies

$$L(v)F_i^h=0, \tag{3.8}$$

where $L(v)$ is Lie derivative with respect to v^h.

The equation (3.8) may be written as

$$L(v)F_i^h=v^t\nabla_t F_i^h-F_i^t\nabla_t v^h+F_t^h\nabla_i v^t=0,$$

from which follows

$$v^t\nabla_t F_{ih}-F_i^t\nabla_t v_h-F_h^t\nabla_i v_t=0.$$

From (2.3), we get

$$v^t\nabla_t F_{ih}+F_i^t\nabla_h v_t-F_h^t\nabla_i v_t=0$$

or

$$v^t\nabla_t F_{ih}-(\nabla_h F_i^t)v_t+(\nabla_i F_h^t)v_t=0$$

by virtue of $F_i^t(\nabla_h v_t)=-(\nabla_h F_i^t)v_t$.

On account of (1.1), (1.4) and (1.5), we have

$$H_{ja}F_i^a=0,$$

that is, $H_{ji}=0$.

Thus we have

THEOREM 3.4. *If* $L(v)F_i^h=0$ *in a nearly Sasakian manifold M, then M is a Sasakian manifold.*

References

1. D.E. Blair, D.K. Shower & K. Yano, *Nearly Sasakian structure*, Kōdai Math. Semi. Rep. **27** (1976).
2. T. Imai, *Notes on semi-symmetric connection*, Tensor, N.S., **27** (1973), 56-58.
3. Z. Olszak, *Five dimensional nearly Sasakian manifolds*. Tensor, N.S., **34** (1980).
4. Z. Olszak, *Nearly Sasakian manifold*, Tensor, N.S., **33** (1979).
5. I. Sato, *On a structure similar to the almost contact structrue*, Tensor.

N. S., **30** (1976).
6. K. Yano, *Differential geometry on a complex & almost complex spaces*, A Pergamon Press Book. (1965).

Kum-Oh Institute of Technology
Kumi, Kyŏngbuk