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# A NOTE ON THE NUMERICAL RANGE OF AN OPERATOR

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## 1. Introduction

The concepts of the numerical range of an operator on a Hilbert space and on a Banach space were introduced by Toeplitz in 1918 and Bauer in 1962 respectively. Bauer's paper was concerned only with finite dimensional Banach spaces, but the concept of numerical range that he introduced is available without restriction of the dimension [1, 2]. In this paper, we define a $C^*$-algebra spatial numerical range of an operator on $C^*$-algebra valued inner product modules introduced by Paschke [4], and give analogous results on these modules as those on Banach spaces.

## 2. Preliminaries and spatial numerical ranges

Let $B$ be a $C^*$-algebra. We denote the action of $B$ on a right $B$-module $X$ by $(x, b) \longrightarrow xb$ $(x \in X,\ b \in B)$. All modules treated below are assumed to have a vector space structure over the complex numbers $\mathbf{C}$ compatible with that of $B$ in the sense that $\lambda(xb) = (\lambda x)b = x(\lambda b)$ $(x \in X,\ b \in B,\ \lambda \in \mathbf{C})$.

DEFINITION 2.1. *A pre-Hilbert B-module* is a right $B$-module $X$ equipped with a conjugate-bilinear map $\langle\,,\rangle : X \times X \longrightarrow B$ (called a $B$-valued inner product on $X$) satisfying;

(i) $\langle x, x\rangle \geqq 0$, $x \in X$ and $\langle x, x\rangle = 0$ only if $x = 0$;
(ii) $\langle x, y\rangle = \langle y, x\rangle^*$, $x$, $y \in X$;
(iii) $\langle xb, y\rangle = \langle x, y\rangle b$, $x, y \in X$, $b \in B$.

For a pre-Hilbert $B$-module $X$ and a symbol $\|\cdot\|_X$ defined by $\|x\|_X = \|\langle x, x\rangle\|^{\frac{1}{2}}$, $\|\cdot\|_X$ is a norm on $X$ [4].

DEFINITION 2.2. *A Hilbert B-module* is a pre-Hilbert $B$-module $X$ which is complete with respect to the norm $\|\cdot\|_X$.

In the rest of this paper, let $B$ be a $C^*$-algebra with a multiplicity identity $e$ and $X$ a Hilbert $B$-module. Note that any $C^*$-algebra $B$ is itself a Hilbert $B$-module with $\langle x, y\rangle = y^*x$ $(x, y \in B)$, and in case $B = \mathbf{C}$, a Hilbert $B$-module is a Hilbert space. Let $B(X)$ be the set of all bounded linear operators on $X$ and $A(X)$ the set of operator $T \in B(X)$ for which there is an operator $T^* \in B(X)$

such that $\langle Tx, y\rangle=\langle x, T^*y\rangle$ for any $x, y\in X$. For each $T\in A(X)$, the adjoint $T^*$ is unique and belongs to $A(X)$. We also denote the operator norm on $B(X)$ by $\|\cdot\|$. It it easy to show that $A(X)$ is a $C^*$-algebra with the involution $T\longrightarrow T^*$ and the norm $\|\cdot\|$ [4]. Let $S(X)$ denote the unit sphere of $X$, $B^*$ the dual space of $B$, and $\prod(x)$ the subset of $X\times B^*$ defined by $\prod(x)=\{(x,f)\in S(X)\times S(B^*) : f(\langle x,x\rangle)=1\}$. Given $x\in S(X)$, we put $D(B,\langle x,x\rangle)=\{f\in S(B^*) : f(\langle x,x\rangle)=1\}$.

DEFINITION 2.3. *The B-spatial numerical range $W_B(T)$ of an operator $T$ of* $A(X)$ is defined by

$$W_B(T)=\{f(\langle Tx,x\rangle) : (x,f)\in\prod(X)\}$$

and *the B-spatial numerical radius* $\omega_B(T)$ of $T$ is the number

$$\omega_B(T)=\sup\{|\lambda| : \lambda\in W_B(T)\}.$$

It is easy to prove that $\omega_B(.)$ is a seminorm on $A(X)$, $\omega_B(T)=\omega_B(T^*)$, $\omega_B(T)\leqq\|T\|$, and

$$W_B(T)=\cup\{\{f(\langle Tx, x\rangle): f\in D(B, \langle x,x\rangle)\} : x\in S(X)\}.$$

REMARK 2.4. $W_B(T)\subset V(A(X), T)$ *where* $V(A(X), T)$ *denotes the numerical range of $T$ of $A(X)$.* For, given $(x, f)\in\prod(X)$, define $F$ on $A(X)$ by $F(S)=f(\langle Sx, x\rangle)$ $(S\in A(X))$. Then $F\in D(A(X), I)$ and so $f(\langle Tx, x\rangle)=F(T)\in V(A(X), T)$ by [1].

LEMMA 2.5. *Let $P$ be a subset of $\prod(X)$ such that its natural projection $\pi_1(P)=\{x: (x, f)\in P$ for some $f\}$ is dense in $S(X)$. Then for each $T\in A(X)$,* $\inf\left\{\frac{1}{\alpha}(\|I+\alpha T\|-1) : \alpha>0\right\}=sup\left\{Re\ f(\langle Tx, x\rangle):(x, f)\in P\right\}$.

*Proof.* Let $\mu=\sup\{\mathrm{Re} f(\langle Tx, x\rangle) : (x, f)\in P\}$. By Remark 2.4 and Theorem 2.5 [1], we have $\mu\leqq\max\{\mathrm{Re}\,\lambda : \lambda\in V(A(X), T)\}=\inf\left\{\frac{1}{\alpha}(\|I+\alpha T\|-1): \alpha>0\right\}=\lim_{\alpha\to0+}\frac{1}{\alpha}\{\|I+\alpha T\|-1\}$. (*) It is obvious when $T=0$, so we assume that $T\neq0$. Choose $\alpha$ such that $0<\alpha<\|T\|^{-1}$. Let $x\in S(X)$ and $\varepsilon>0$. Since $\pi_1(P)$ is dense in $S(X)$, there exists $(y, g)\in P$ such that $\|x-y\|_X<\varepsilon$. We have $\mathrm{Re}\, g(\langle Ty, y\rangle)\mu\leqq\|T\|$ and so $\|(I-\alpha T)y\|_X\geqq\mathrm{Re}\ g(\langle(I-\alpha T)y, y\rangle=1-\alpha\ \mathrm{Re}\ g(\langle Ty, y\rangle)\geqq1-\alpha\mu>0$. Therefore $\|(I-\alpha T)x\|_X\geqq1-\alpha\mu-\|I-\alpha T\|\varepsilon$. Since $\varepsilon$ is arbitrary, this gives $\|(I-\alpha T)x\|_X\geqq1-\alpha\mu$, and therefore $\|(I-\alpha T)x\|_X\geqq(1-\alpha\mu)\|x\|_X$ $(x\in X)$. If we replace $x$ by $(I+\alpha T)x$, this gives

$\|(I+\alpha T)x\|_X\leqq\frac{1}{1-\alpha\mu}\|(I-\alpha^2T^2)x\|_X$. $(x\in X)$, and so

$\|I+\alpha T\|\leqq\frac{1+\alpha^2\|T^2\|}{1-\alpha\mu}$. Therefore $\frac{1}{\alpha}\{\|I+\alpha T\|-1\}\leqq\frac{\mu+\alpha\|T^2\|}{1-\alpha\mu}$, and this with (*) completes the proof.

By Lemma 2.5 and Theorem 2.5 [1], we have

$\sup\{\mathrm{Re}\, f(\langle Tx,\ x\rangle) : (x,\ f)\in P\} = \sup\{\mathrm{Re}\,\lambda : \lambda\in V(A(X),\ T)\}$ where $P$ denotes a subset of $\prod(X)$ such that its natural projection $\pi_1(P)$ is dense in $S(X)$. Also $V(A(X),\ T)$ is a closed convex set [1]. Thus we obtain the following results;

THEOREM 2. 6. *Let $P$ be a subset of $\prod(X)$ such that its natural projection $\pi_1(P)$ is dense in $S(X)$. Then for each $T\in A(X)$, $\overline{co}\{f(\langle Tx,\ x\rangle) : (x,\ f)\in P\} = V(A(X),\ T)$ where $\overline{co}\ E$ denotes the closed convex hull of a set $E$.*

COROLLARY 2.7. *For each $T\in A(X)$, we have*

(i) $\overline{co}\ W_B(T) = V(A(X),\ T)$

(ii) $\omega_B(T) = \sup\{|\lambda| : \lambda\in V(A(X),\ T)\}$.

By Theorem 2.6 [1], we have $Sp(T)\subset V(A(X),\ T = \overline{co}W_B(T)$ where $Sp(T)$ denotes the spectrum of $T$.

## 3. Topological properties

DEFINITION 3.1. *The norm$\times$weak* topology in $X\times B^*$* is the product topology in $X\times B^*$ given by the norm topology on $X$ and the weak* topology on $B^*$.

The following two results are essentially due to Bonsall, Cain and Schneider [1].

LEMMA 3.2. *Let $\pi_1$ denote the natural projection of $X\times B^*$ onto $X$, and let $E$ be a subset of $\prod(X)$ that is relatively closed in $\prod(X)$ with respect to the norm$\times$ weak* topology. Then $\pi_1(E)$ is a (norm) closed subset of $X$.*

*Proof.* Let $x_n\in\pi_1(E)$ and $x_n\longrightarrow x\in X$. Then there exist elements $f_n$ of $S(B^*)$ such that $(x_n,\ f_n)\in E$. By the weak* compactness of the closed unit ball in $B^*$, there exists a weak* cluster point $f$ of the sequence $\{f_n\}$ with $\|f\|\leqq 1$.
We have $f(\langle x,\ x\rangle) = (f-f_n)(\langle x,\ x\rangle) + f_n(\langle x-x_n,\ x_n\rangle) + f_n(\langle x_n,\ x_n\rangle) + f_n(\langle x,\ x-x_n\rangle)$, and so $|f(\langle x,\ x\rangle)-1| \leqq |(f-f_n)(\langle x,\ x\rangle)| + |f_n(\langle x-x_n,\ x_n\rangle)| + |f_n(\langle x,\ x-x_n\rangle)| \leqq |(f-f_n)(\langle x,\ x\rangle)| + \|x-x_n\|_X + \|x-x_n\|_X\|x\|_X$. Since $|(f-f_n)(\langle x,\ x\rangle)|$ and $\|x-x_n\|_X$ are arbitrarily small for all sufficiently large $n$, it follows that $f(\langle x,\ x\rangle)=1$ and therefore $(x,\ f)\in\prod(X)$. But $E$ is relatively closed in $\prod(X)$ for the norm $\times$ weak* topology, and so $(x,\ f)\in E$ and $x\in\pi_1(E)$.

THEOREM 3.3. *$\prod(X)$ is a connected subset of $X\times B^*$ with the norm$\times$weak* topology, unless $X$ has dimension one over $R$.*

*Proof.* Suppose that $\prod(X) = E\cup F$ where $E$, $F$ are relatively closed in $\prod(X)$ for the norm$\times$weak* topology, and $E\cap F=\phi$. By Lemma 3.2, $\pi_1(E)$ and $\pi_1(F)$ are norm closed subsets of $X$ and $\pi_1(E)\cup\pi_1(F) = S(X)$. Suppose that $x\in\pi_1(E)\cap\pi_1(F)$. Then there are $f$ and $g$ in $B^*$ such that $(x,\ f)\in E$ and $(x,\ g)\in F$. For $t$ with $0\leqq t\leqq 1$, we have $(tf+(1-t)g)(\langle x,\ x\rangle) = tf(\langle x,\ x\rangle) + (1-t)g(\langle x,\ x\rangle) = 1$ and hence

$$\|tf+(1-t)g\| \geqq 1 \qquad (0 \leqq t \leqq 1).$$

Also $\|tf+(1-t)g\| \leqq t\|f\|+(1-t)\|g\|=1$. Thus we have shown that $(x, tf+(1-t)g) \in \prod(X)$ $(0 \leqq t \leqq 1)$ which is impossible since $E \cap F=\phi$. Therefore $\pi_1(E) \cap \pi_1(F)=\phi$. Now if $X$ doesn't have dimension one over $R$, then the set $S(X)$ is connected. Thus we must have $\pi_1(E)=\phi$ or $\pi_1(F)=\phi$. Therefore $\prod(X)$ is connected.

COROLLARY 3.4. *$W_B(T)$ is connected unless $X$ has dimension one over $R$.*

*Proof*. We have $|f(\langle Tx, x\rangle)-g(\langle Ty, y\rangle)| \leqq \|Tx-Ty\|_X+\|Ty\|_X\|x-y\|_X+|(f-g)(\langle Ty, y\rangle)|$ $((x, f), (y, g) \in \prod(X))$. Therefore the mapping $(x, f) \rightarrow f(\langle Tx, x\rangle)$ is a continuous mapping of $\prod(X)$ with the relative norm$\times$weak* topology onto $W_B(T)$. Therefore by Theorem 3.3, $W_B(T)$ is connected, unless $X$ has dimension one over $R$.

## References

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